HATCHER'S ALGEBRAIC TOPOLOGY SOLUTIONS

REID MONROE HARRIS

Van Kampen's Theorem

Problem 1. Suppose G and H are nontrivial groups. Suppose $x = g_1 h_1 \cdots g_n h_n$ lies in the center of $G * H$, where $g_i \in G$ and $h_i \in H$. For any $g \in G * 1_H$, we have

$$g g_1 h_1 \cdots g_n h_n g^{-1} h_n^{-1} g_n^{-1} \cdots h_1^{-1} g_1^{-1} = 1.$$

The only way for this to be true for all g is if $h_i = 1_H$ for all i. Therefore, $x \in G * 1_H$. For any nontrivial $h \in 1_G * H$, we see that $[x, h] \neq 1$ unless $x = 1$. Therefore, the center of $G * H$ is trivial.

Problem 2.

Problem 3.

Problem 4.

Problem 5.

Covering Spaces

Problem 1. Let $p : \tilde{X} \to X$ be a covering space and $A \subset X$ and $\tilde{A} = p^{-1}(A)$. We want to show that $p : \tilde{A} \to A$ is a covering space. Let $x \in A$. Then there is a U open neighborhood of x in X such that $p^{-1}(U)$ is a disjoint union of open sets each of which are homeomorphic to U via p (evenly covered). We want to show that $U \cap A$ is evenly covered by $p|\tilde{A}$. Let $p^{-1}(U) = \bigsqcup U_\alpha$. Then

$$p^{-1}(U \cap A) = p^{-1}(U) \cap \tilde{A} = \left(\bigsqcup U_\alpha\right) \cap \tilde{A} = \bigsqcup (U_\alpha \cap \tilde{A}).$$

Now, it suffices to show that each $U_\alpha \cap \tilde{A}$ is homeomorphic to $U \cap A$ via $p|\tilde{A}$. But, $p : U_\alpha \to U$ is a homeomorphism and so $p|(U_\alpha \cap A)$ is a homeomorphism onto its image $U \cap A$, each with the subspace topology.

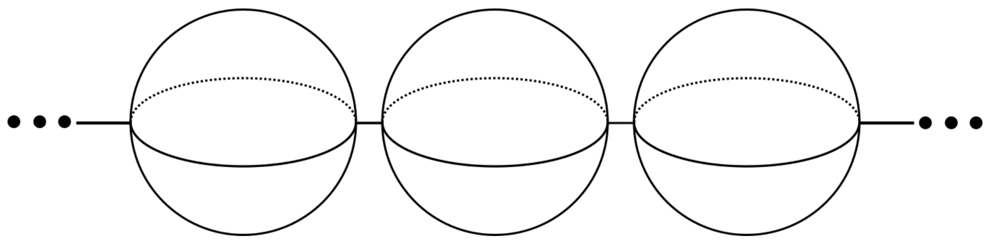
Problem 2. Let $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ be covering spaces. We show that $p_1 \times p_2$ is a covering map. Let $(x_1, x_2) \in X_1 \times X_2$. Then, there are open sets $U_i \subset X_i$ containing x_i that satisfy the covering condition on p_i, for $i = 1, 2$. We show that $U_1 \times U_2$ is evenly covered by $p_1 \times p_2$. We have

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = p_1^{-1}(U_1) \times p_2^{-1}(U_2) = \left(\bigsqcup_\alpha U_{1,\alpha}\right) \times \left(\bigsqcup_\beta U_{2,\beta}\right) = \bigsqcup_{\alpha,\beta} (U_{1,\alpha} \times U_{2,\beta}).$$

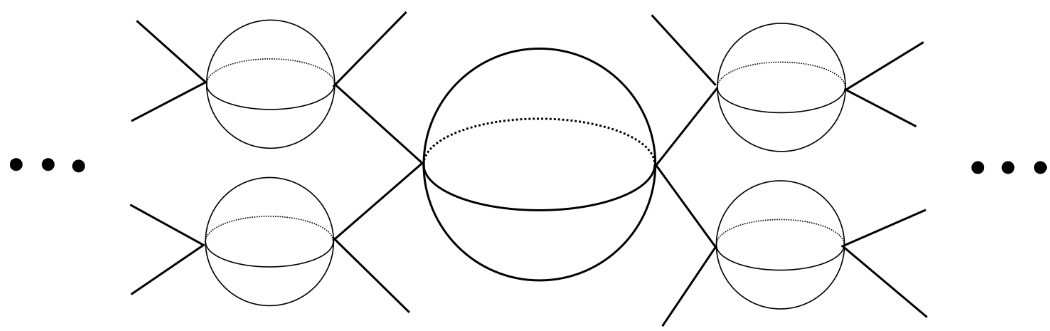
It thus suffices to show that each $U_{1,\alpha} \times U_{2,\beta}$ is mapped homeomorphically onto $U_1 \times U_2$, but this follows since $p_1 \times p_2 : U_{1,\alpha} \times U_{2,\beta} \to U_1 \times U_2$ is a product of homeomorphisms.

Problem 3. Let $p : \tilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for each $x \in X$. If $\tilde{X}$ is compact Hausdorff, we immediately get that $X = p(\tilde{X})$ is compact. If $x, y \in X$, let $U_x, U_y \subset X$ be constructed as follows. The Hausdorff condition extends to finite sets of points, so we can find disjoint open neighborhoods $\tilde{U}_x$ and $\tilde{U}_y$ of $p^{-1}(x)$ and $p^{-1}(y)$, respectively. By taking even smaller open sets, we can assume that $\tilde{U}_x$ and $\tilde{U}_y$ evenly cover their images U_x and U_y. We claim that these two sets are open and disjoint. They are open since p is an open map. Suppose $z \in U_x \cap U_y$. Then there is a lift $\tilde{z}$ of z such that $\tilde{z} \in \tilde{U}_x = p^{-1}(U_x)$. But then, $\tilde{z} \in p^{-1}(U_y) = \tilde{U}_y$, which contradicts the fact that $\tilde{U}_x$ and $\tilde{U}_y$ are disjoint.

Problem 4. The simply connected covering of the space consisting of a sphere and one of its diameters is shown below.

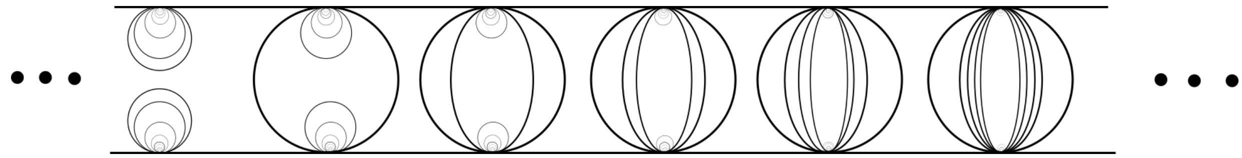


The simply connected covering of a space consisting of a sphere intersected by a circle in two points is shown below.



Problem 5. Let A denote the left edge $\{0\} \times [0, 1]$ of the space X. There is a neighborhood of $(0, 0) \in A$ lifts homeomorphically to $\tilde{X}$. Let t^* be the supremum of all t such that this lift extends over a neighborhood of $\{0\} \times [0, t]$. If $t^* < 1$, then we can find a neighborhood of t^* and a lift on that neighborhood and agrees on the intersection with the set $\{0\} \times [0, t^*)$. Hence, we can find a lift over a neighborhood of $\{0\} \times [0, t^* + \epsilon]$. Therefore, $t^* = 1$.

If $[0, 1/N] \times [0, 1]$ is contained in this neighborhood, then the loop γ which runs up $\{0\} \times [0, 1]$, across $[0, 1/N] \times \{1\}$, down $\{1/N\} \times [0, 1]$ and back across $[0, 1/N] \times \{0\}$ is contained in this neighborhood as well and so lifts to a loop in $\tilde{X}$. This lift must be nontrivial in $\pi_1(\tilde{X}, 0)$ since it projects to something nontrivial. Hence, $\pi_1(\tilde{X}, 0) \neq 0$.



Problem 6. We have the following 2-sheeted covering space Y of $\tilde{X}$:

Consider a connected neighborhood U of the vertex v in the Hawaiian earring X. Taking the preimage of U under the composition $Y \to \tilde{X} \to X$, we get that far to the right of the diagram above, there is a connected component of U which contains a larger loop that is tangent to both the top and bottom lines, hence contains two preimages of the vertex and cannot be homeomorphic to U via $Y \to X$. Therefore, the composition $Y \to \tilde{X} \to X$ is not a covering space.

However, if $p_1 : Y \to \tilde{X}$ is a covering and $p_2 : \tilde{X} \to X$ is a covering, then the composition $p_3 : Y \to X$ satisfies the unique path lifting property. Indeed, let γ be a path in X and let $y \in Y$ be in the preimage $p_3^{-1}(\gamma(0))$. There is a unique path lifting $\tilde{\gamma}$ in $\tilde{X}$ of γ beginning at $p_1(y)$ and there is a unique path lifting $\tilde{\gamma}$ over p_1 beginning at y. The result is a unique path which lifts γ over p_3.

Problem 7. Let Y and $f : Y \to \mathbb{S}^1$ be as in the problem statement. Assume f has a lift $\tilde{f} : Y \to \mathbb{R}$. Let $v \in Y$ be the point $(0, 0)$. Consider the path $\gamma : [0, 1) \to Y$ defined by traversing Y counterclockwise beginning at v and $\gamma(t)$ tends to the vertical line $[-1, 1]$ as $t \to 1$. This defines a path $f \circ \gamma$ given by traversing once around the circle which has a lift $\tilde{f} \circ \gamma$ to $\mathbb{R}$. However, for every neighborhood U of $\gamma(0)$ and for every $T \in [0, 1)$, there is a $T \le t_0 < 1$ such that $\gamma(t_0) \in U$. If such a lift existed, then $\tilde{f} \circ \gamma(0) = 0$ and $\tilde{f} \circ \gamma(t) \to 1$ as $t \to 1$ which contradicts the assumption that $\tilde{f}$ is continuous.

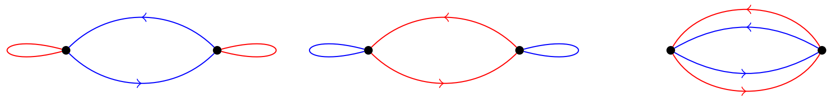
Problem 8. Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be simply connected covering spaces of path connected, locally path connected spaces X and Y. Suppose $X \simeq Y$ and let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. We use the lifting criterion to construct lifts $F : \tilde{X} \to \tilde{Y}$ and $G : \tilde{Y} \to \tilde{X}$ such that $f \circ p = q \circ F$ and $g \circ q = p \circ G$. Using these two relations, we get that $q \simeq q \circ F \circ G$ and $p \simeq p \circ G \circ F$. From the homotopy lifting property, $G \circ F$ is homotopic to a deck transformation, hence we can compose with the deck transformation to get $G' \circ F \simeq \mathrm{id}_{\tilde{Y}}$ for some map $G' : \tilde{Y} \to \tilde{X}$. Likewise, there is some $G'' : \tilde{X} \to \tilde{Y}$ such that $F \circ G'' \simeq \mathrm{id}_{\tilde{X}}$. It is straightforward to show that $G' \simeq G''$:

$$G' \simeq G' \circ (F \circ G'') \simeq (G' \circ F) \simeq G'' \simeq G''.$$

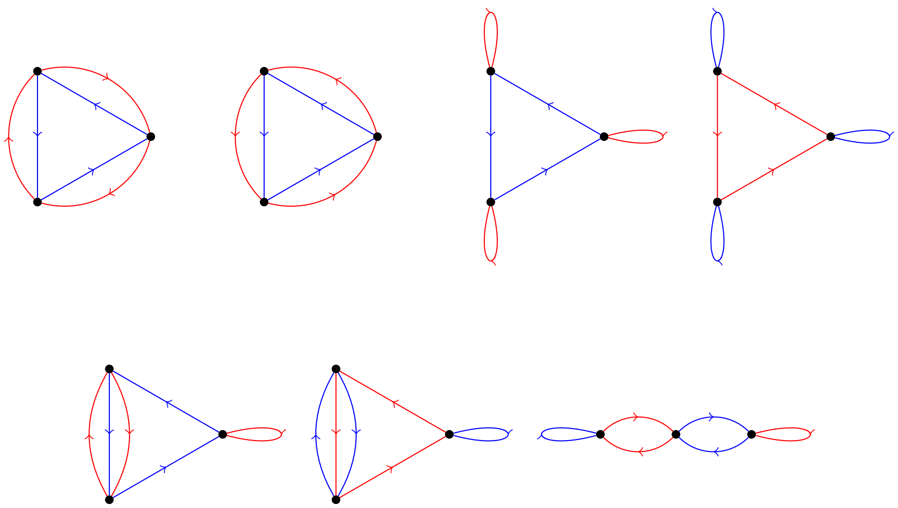
Hence, $\tilde{X} \simeq \tilde{Y}$.

Problem 9. Suppose X is path connected and locally path connected and $\pi_1(X)$ is finite. Let $f : X \to \mathbb{S}^1$. This induces a homomorphism $f_* : \pi_1(X) \to \mathbb{Z}$, hence must be the zero map. By the lifting criterion, f lifts to a map $\tilde{f} : X \to \mathbb{R}$ which is nullhomotopic, hence f is nullhomotopic.

Problem 10. Since any index-2 subgroup is normal, the connected 2-sheeted covering spaces are classified by the surjective homomorphisms $F_2 = \langle a, b \rangle \to \mathbb{Z}/2\mathbb{Z}$. There are exactly 3 such homomorphisms given by $a \mapsto 1, b \mapsto 0$ and $a \mapsto 0, b \mapsto 1$ and $a, b \mapsto 1$. These are shown below.

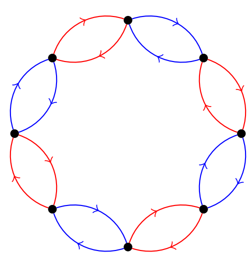


The following seven 3-sheeted coverings of $\mathbb{S}^1 \vee \mathbb{S}^1$ are found using combinatorial means.

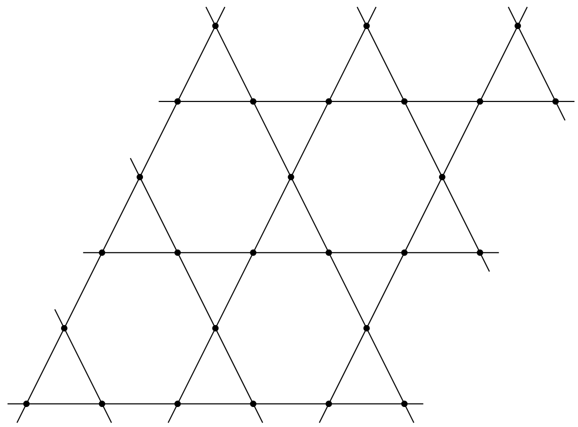


Problem 11. For this, we take the infinite 3-valence tree $\tilde{X}$ which covers the two graphs described as follows: X is the graph obtained by connecting two disjoint circles with a line segment and Y is the graph obtained from X by connecting a third disjoint circle to the other two circles via line segments (for a total of 6 vertices). Then, $\tilde{X}$ is the universal cover of both X and Y, but X does not cover any other graph (there is no graph with a single 3-valence vertex) and Y does not cover X. This can be seen since no three of the vertices of Y are the vertices of a triangle lying in Y (the circle attached to either vertex must lift to a triangle).

Problem 12. Let $\tilde{X}$ be the graph shown below. The red and blue edges indicate the two oriented edges of $\mathbb{S}^1 \vee \mathbb{S}^1$.



Consider the basepoint $\tilde{x}_0$ of $\tilde{X}$ to be the topmost vertex. The subgroup N corresponding to this covering contains a^2, b^2, and $(ab)^4$. The normal subgroup generated by these three elements is the kernel of the homomorphism $F_2 \to D_4$, hence has index 8. Since $\tilde{X}$ is an 8-sheeted covering space, N is normally generated by a^2, b^2, and $(ab)^4$.



Problem 13. Let N denote the subgroup of F_2 generated by the cubes of all elements. The graph of the covering space $\tilde{X}$ corresponding to N is shown below (embedded in a torus).

To see that this is indeed the covering space corresponding to the given normal subgroup, note that the generators each correspond to loops in $\tilde{X}$, so N is contained in the normal subgroup corresponding to $\tilde{X}$. On the other hand, $\tilde{X}$ is a 27-fold cover (the same as $[F_2 : N]$) so we obtain equality.

Problem 14. The covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ are classified by the conjugacy classes of subgroups of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a, b | a^2 = b^2 = 1\rangle$. The conjugacy classes of subgroups of this group have representatives exactly the subgroups generated by $\{ab\}$, $\{a\}$, $\{b\}$ and 0. The covering space corresponding to the trivial subgroup is an infinite "string of spheres". The subgroups normally generated by a and b correspond to the two covering spaces $\mathbb{RP}^2 \vee \mathbb{S}^2 \vee \mathbb{RP}^2 \to \mathbb{RP}^2 \vee \mathbb{RP}^2$. The subgroup $\langle ab\rangle$ corresponds to the covering space X consisting of two spheres intersecting in exactly two antipodal points.

Problem 15. Let $p : \tilde{X} \to X$ be a simply connected covering space of X and let $A \subset X$ be a path connected, locally path connected subspace, with $\tilde{A} \subset \tilde{X}$ a path-component of $p^{-1}(A)$. We wish to show that $p_*(\pi_1(\tilde{A}))$ is the kernel of the inclusion $\pi_1(A) \to \pi_1(X)$. First, if $\gamma \in \pi_1(\tilde{A})$, then $p(\gamma)$, as a path in X, is contractible since it lifts to a loop in the universal cover $\tilde{X}$. Thus, $p_*(\pi_1(\tilde{A}))$ is contained in the kernel. Suppose $\gamma \in \pi_1(A)$ is a loop in A which is contractible in X. Then, γ lifts to the universal cover $\tilde{X}$, hence must lift to a loop $\tilde{\gamma}$ in $\tilde{A}$, since $\tilde{A}$ is path-connected. Hence, $\gamma = p_*(\tilde{\gamma})$.

Problem 16. Let $X \to Y \to Z$ be maps such that $p : Y \to Z$ and $q : X \to Z$ are covering spaces and Z is locally path connected. We wish to show that $\pi : X \to Y$ is a covering space. Let $y \in Y$. There is a neighborhood U of y such that the restriction $p : U \to p(U)$ is a homeomorphism. By taking a smaller neighborhood, we can assume that $p(U)$ is path connected and also evenly covered by q. Then, the restriction $q : p^{-1}(U) \to U \to p(U)$ is a covering space. Hence, $p^{-1}(U) = q^{-1}(p(U))$ is a disjoint union of sets each of which maps homeomorphically onto $p(U)$ via q and hence homeomorphically onto U via π.

Problem 17. Let G be a group and N a normal subgroup. Let $G = \langle g_\alpha | r_\beta\rangle$. We can build a space X with $\pi_1(X) = G$ as follows. Take bouquet of circles, one circle for each generator and glue on 2-cells as prescribed via the relations r_β. Then, X is path connected and locally path connected. By the Galois correspondence, there is a normal covering space $\tilde{X} \to X$ such that $\pi_1(\tilde{X}) = N$ and $G(\tilde{X}) \cong G/N$.

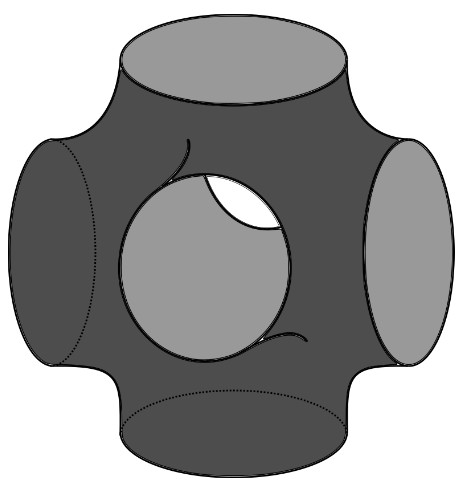
Problem 18. Let X be path connected, locally path connected and semilocally simply connected. Call a path connected covering $\tilde{X} \to X$ *abelian* if it is a normal covering space with abelian deck transformation group. We want to construct the *universal abelian cover*. Suppose $\tilde{X} \to X$ is the universal abelian cover corresponding to the normal subgroup $N < G = \pi_1(X)$. Now, G/N is abelian, so the quotient map $G \to G/N$ factors as $G \to G/[G,G] \to G/N$. Hence, $[G,G] < N$. The universal property states that N is the smallest normal subgroup of G whose quotient group is abelian. Since $[G,G]$ is normal ($k[g,h]k^{-1} = [kgk^{-1}, khk^{-1}]$), this implies that $N = [G,G]$. Note that for any other abelian cover $Y \to X$ with normal subgroup K, we have $[G,G] < K$, so that $\tilde{X}$ covers Y.

For $X = \mathbb{S}^1 \vee \mathbb{S}^1$, we have that $\pi_1(X) = F_2$ which has abelianization $\mathbb{Z}^2$ and commutator subgroup $[G,G]$. Taking the quotient of the universal cover $\tilde{X}/[G,G]$ gives us a graph with vertices on $\mathbb{Z}^2$ and edges connecting adjacent horizontal and vertical vertices oriented upward and to the right.

For $X = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$, the universal abelian cover is similarly realized as a subgraph of $\mathbb{R}^3$ with vertices at $\mathbb{Z}^3$ and edges connecting adjacent vertices oriented upward, to the right and inward.

Problem 19. First, the abelianization of $\pi_1(M_g) = G$ is $\mathbb{Z}^{2g}$. If $n \leq 2g$, then there is a normal subgroup of G containing the commutator subgroup and with quotient group $\mathbb{Z}^n$. Hence, there is a covering of M_g with deck transformation group isomorphic to $\mathbb{Z}^n$. However, if $n > 2g$, assume that such a cover exists with normal subgroup N. Then the quotient map $G \to G/N$ factors through $\mathbb{Z}^{2g} \to G/N = \mathbb{Z}^n$ and is surjective, an impossibility.

The case for $n = 3$ and $g = 3$ is shown below for a fundamental domain. The covering space in this case is obtained from this domain by attaching this domain to each unit cube with integer vertices in $\mathbb{R}^3$. The covering space for $n = 3$, $g \geq 3$, is obtained by



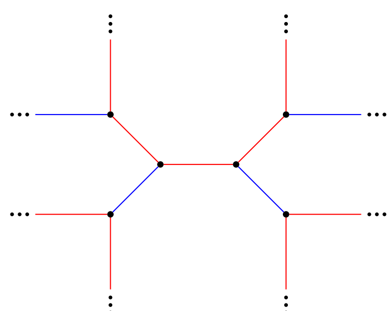
adding a genus into the fundamental domain above. Suppose such a covering space $\tilde{X}$ in $\mathbb{R}^3$ exists. Hence, we have an embedding $\tilde{X} \to \mathbb{R}^3$ with deck transformation group $\mathbb{Z}^3$. Taking the quotient gives an embedding $M_g \to \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ into the 3-torus which induces a surjection $\pi_1(M_g) \to \pi_1(\mathbb{T}^3) = \mathbb{Z}^3$ (deck transformation action). Conversely, if we are given an embedding $\phi : M_g \to \mathbb{T}^3$, we let $\tilde{M}_g$ be the cover corresponding to the kernel of the surjection $\pi_1(M_g) \to \pi_1(\mathbb{T}^3)$. Then, we can lift (lifting criterion) to get a map $\Phi : \tilde{M}_g \to \mathbb{R}^3$. This is injective: if $\Phi(x) = \Phi(y)$, then

$$\phi \circ \pi'(x) = \pi \circ \Phi(x) = \pi \circ \Phi(y) = \phi \circ \pi'(y),$$

so $\pi'(x) = \pi'(y)$ and $x = n + y$ for some $n \in \mathbb{Z}^3$. Then n must be 0 since $\Phi(x) = \Phi(x + n) = \Phi(x) + n$. This is a homeomorphism onto its image since it is a lift of an embedding.

Problem 20. The fundamental group of the Klein bottle is $\mathbb{Z} \rtimes \mathbb{Z}$. The subgroups $\mathbb{Z} \times 2\mathbb{Z}$ and $\mathbb{Z} \rtimes 3\mathbb{Z}$ are nonnormal subgroups corresponding to covers by the torus and Klein bottle, respectively.

Problem 21. Let X be the space obtained by gluing a Möbius band M along its boundary to the latitude circle of the torus $\mathbb{T}^2$. The universal cover is the product $T \times \mathbb{R}$ where T is the valence 3 tree shown below:



The action of

$$\pi_1(X) = \pi_1(\mathbb{T}^2) *_N \pi_1(M) = \langle a, b, c \mid ab = ba, c^2 = a \rangle = \langle b, c \mid bc^2 = c^2 b \rangle$$

is given as follows. The element c acts by flipping T over a midpoint of a selected blue edge and translating along the $\mathbb{R}$ factor 1 unit. The element b acts by translating T along the red direction by 1 unit.

Problem 22. Let G_1 and G_2 act on X_1 and X_2 via a covering space action. We show that the action of $G_1 \times G_2$ on $X_1 \times X_2$ is a covering space action. Let $(x_1, x_2) \in X_1 \times X_2$ and let U_1 and U_2 be neighborhoods of x_1 and x_2 which satisfy the covering space condition, i.e. $gU_1 \cap U_1 = \varnothing$ and $hU_2 \cap U_2 = \varnothing$ for all $g \in G_1$ and $h \in G_2$. Then,

$$(h, g) \cdot (U_1 \times U_2) \cap (U_1 \times U_2) = (gU_1 \times hU_2) \cap (U_1 \times U_2) = \varnothing.$$

Now, we define $\varphi : (X_1 \times X_2)/(G_1 \times G_2) \to X_1/G_1 \times X_2/G_2$ by $(G_1 \times G_2) \cdot (x_1, x_2) \mapsto (G_1 x_1, G_2 x_2)$. This is well-defined since if (x_1', x_2') is in the same orbit as (x_1, x_2) then x_1' and x_2' are in the same orbit as x_1 and x_2, respectively. The inverse of this map takes $(G_1 x_1, G_2 x_2) \mapsto (G_1 \times G_2) \cdot (x_1, x_2)$, which is well-defined and continuous. Hence, φ is an homeomorphism.

Problem 23. Let $x \in X$, where X is Hausdorff and suppose G acts on X freely and properly discontinuously. Let U be a neighborhood such that $G' = \{g \in G : gU \cap U \neq \varnothing\}$ is finite. For each $g \in G'$, we have the neighborhood gU of gx. Let U' and V' be disjoint neighborhoods of x and gx. Then $\tilde{U} = U' \cap g^{-1}V'$ is a neighborhood of x such that $g\tilde{U} \cap \tilde{U} = \varnothing$. We repeat this for every $g \in G'$ to get a neighborhood U of x such that $gU \cap U$ is empty for each $g \in G$. Hence, the G-action is a covering space action on X.

Problem 24. a) Let $X \xrightarrow{\pi} Y \xrightarrow{f} X/G$ be a sequence of covering spaces. Let H be the set of elements $g \in G$ such that $\pi(gx) = \pi(x)$ for each $x \in X$. This set is nonempty since it contains the identity element and is a subgroup. We want to show that $Y \to X/G$ is isomorphic to the covering space $X/H \to X/G$. If $\pi(x) = \pi(x')$, then there is an element $g \in G$ such that $gx = x'$. Then, by definition, $g \in H$. Hence, we have a well-defined map $Y \to X/H$ obtained by sending $y \in Y$ to the orbit xH of any point x in its fiber. This has the inverse image $xH \mapsto \pi(x)$. This is a covering transformation since the composition $xH \mapsto \pi(x) \mapsto (f \circ \pi)(x) = xG$ is exactly the covering map $X/H \to X/G$.

b) The covers $X/H_1, X/H_2 \to X/G$ are isomorphic hence correspond to conjugate subgroups N_1, N_2 of $\pi_1(X/G)$, say $\tilde{g}N_1\tilde{g}^{-1} = N_2$, for $\tilde{g} \in \pi_1(X/G)$. We have a surjective homomorphism $\pi_1(X/G) \to G$ given by the action by deck transformations on $X \to X/G$. We claim that the images of N_1 and N_2 under this surjection are H_1 and H_2. We show this for H_1. If $h \in H_1$, then there is a path $\tilde{\gamma}$ from x_0 to hx_0 in X. This path projects to a loop γ in X/G, which defines a homotopy class $[\gamma]$. Under the surjection above, this homotopy class has image h (since deck transformations are determined by their action at a single point). If $[\gamma] \in N_1$, then γ comes from a loop γ' in X/H_1. The lift $\tilde{\gamma}$ to X then determines a deck transformation which is a deck transformation of the covering space $X \to X/H_1$ which proves the claim. If g is the image of $\tilde{g}$ under this surjection, then $gH_1g^{-1} = H_2$.

Conversely, if $gH_1g^{-1} = H_2$ for some element $g \in G$, then we can define an isomorphism of covering spaces $\Phi : X/H_1 \to X/H_2$ by taking $H_1x \mapsto H_2gx$. This map is well-defined exactly because of the conjuation between H_1 and H_2 and it has an obvious inverse.

c) If H is normal in G, then for arbitrary elements Hx, Hgx in the fiber over Gx, we have $Hgx = gHx$, where $g \in G$ is a deck transformation on $X \to X/G$. By normality, this descends to a deck transformation of $X/H \to X/G$. Hence, this cover is normal.

Conversely, if the cover is normal, then it corresponds to a normal subgroup N of $\pi_1(X/G)$, which surjects onto H via the surjective homomorphism $\pi_1(X/G) \to G$. Hence, H is normal in G.

Problem 25. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\varphi(x, y) = (2x, y/2)$ and let G be the group generated by φ and $X = \mathbb{R}^2 \setminus 0$. Let $(x, y) \in X$ and let U be the neighborhood $(3x/4, 5x/4) \times (3y/4, 5y/4)$. Then, for any n,

$$\varphi^n(U) = (3 \cdot 2^{n-2}x, 5 \cdot 2^{n-2}x) \times (3 \cdot 2^{-n-2}y, 5 \cdot 2^{-n-2}y).$$

If $(x', y') \in U \cap \varphi^n(U)$, then $|x' - x| < x/4$ and $|y' - y| < x/4$ but also, $|x' - 2^n x| < 2^{n-2}x$ and $|y' - 2^{-n}y| < 2^{-n-2}x$. Hence,

$$|x - 2^n x| < (1 + 2^n)x/4 = \text{ and } |y - 2^{-n}y| < (1 + 2^{-n})y/4.$$

These imply that $|1-2^n| < (1+2^n)/4$ and $|1-2^n| < (1+2^{-n})/4$. If $n > 0$, then the equation on the right fails and if $n < 0$, then the equation on the left fails. Hence, this action is a covering space action.

Let $(1,1)$ be the basepoint of X. We get an exact sequence

$$0 \to \pi_1(X) \to \pi_1(X/G) \to G \to 0.$$

This sequence splits since we have a map $\rho : G \to \pi_1(X/G)$ given by taking $g \in G$ to the projection of the straight line from $(1,1)$ to $g \cdot (1,1)$. Also, the subgroups $\rho(G)$ and $\pi_1(X)$ commute so that $\pi_1(X/G) \cong \mathbb{Z} \times \mathbb{Z}$.

The quotient space X/G is not Hausdorff since the orbit of $(1,0)$ contains $(2^n,0)$ and the orbit of $(1,1)$ contains $(2^n,2^{-n})$ for every n. The distance between these two points is 2^{-n}, hence every pair of open neighborhoods of $(1,0)$ and $(1,1)$ intersect.

Problem 26. Let $p : \tilde{X} \to X$ be a covering space and let X be connected, locally path connected, and semilocally simply connected.

a) Let $\mathcal{C}$ denote the connected components of $\tilde{X}$. We define a map $p^{-1}(x_0) \to \mathcal{C}$ by sending a point $\tilde{x}$ in the fiber to the connected component $\tilde{C}(\tilde{x})$ containing it. If $[\gamma] \in \pi_1(X,x_0)$, then $[\gamma]\cdot\tilde{x}$ is the endpoint of the lift of γ beginning at $\tilde{x}$, hence, $\tilde{C}([\gamma]\cdot\tilde{x}) = \tilde{C}(\tilde{x})$ and the map descends to a map $C : p^{-1}(x_0)/\pi_1(X,x_0) \to \mathcal{C}$. The inverse of this map takes a component $C \in \mathcal{C}$ to the intersection $C \cap p^{-1}(x_0)$. If $\tilde{x}_0, \tilde{x}_1 \in C \cap p^{-1}(x_0)$, then there is a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$. This descends to a loop in X which defines a homotopy class $[\gamma] \in \pi_1(X,x_0)$. Then, $[\gamma]\cdot\tilde{x}_0 = \tilde{x}_1$, so that this is indeed a map $\mathcal{C} \to p^{-1}(x_0)/\pi_1(X,x_0)$ which is the inverse of C above.

b) Let C denote the component of $\tilde{X}$ containing a given lift $\tilde{x}_0$ of x_0 and let N denote the subgroup corresponding to the connected covering space $C \to X$. Let $H = \mathrm{Stab}_{\tilde{x}_0}(\pi_1(X,x_0))$ be the stabilizer of $\tilde{x}_0$ for the action of $\pi_1(X,x_0)$ on $p^{-1}(x_0)$, i.e. the subgroup of all $[\gamma] \in \pi_1(X,x_0)$ such that $[\gamma]\cdot\tilde{x}_0 = \tilde{x}_0$. First, N is the set of all classes $[\gamma] \in \pi_1(X,x_0)$ which lift to a loop in the cover C at $\tilde{x}_0$. Therefore, $[\gamma]$ acts on $p^{-1}(x_0)$ by stabilizing $\tilde{x}_0$ and so $N < H$. For the other direction, if $[\gamma]$ stabilizes $\tilde{x}_0$, then it lifts to a loop based at $\tilde{x}_0$, by definition. Hence, $H = N$.

Problem 27. For a universal cover $p : \tilde{X} \to X$, we have the two given actions of $\pi_1(\tilde{X},x_0)$ on $p^{-1}(x_0)$ given by the restriction of the deck transformations and the monodromy action. Suppose $X = \mathbb{S}^1 \vee \mathbb{S}^1$. Let $[\gamma],[\eta] \in G = \pi_1(X,e)$, where e is the vertex. If $[\gamma]$ and $[\eta]$, then the composite $[\gamma\cdot\eta]$ acts on $p^{-1}(e)$ by taking a point x in the fiber along a lift of γ based at x, then along a lift of η based at $[\gamma]\cdot x$. Hence, $[\gamma\cdot\eta]\cdot x = [\eta]\cdot([\gamma]\cdot x)$, i.e. the monodromy action is a right action. However, the deck transformation group acts via a left action. Therefore, these actions are not the same.

If $X = \mathbb{S}^1 \times \mathbb{S}^1$, then the monodromy action and the deck transformation group are the same as we now show.

The monodromy action and deck transformation action on a fiber $p^{-1}(x_0)$ are equivalent if $\pi_1(X,x_0)$ is abelian. To show this, we note that both actions are now left actions and we need only show that given the monodromy action of $[\gamma] \in \pi_1(X,x_0)$ on the fiber, it extends to a deck transformation of the universal cover. Let $y_0 \in X$ and let η be any path from x_0 to y_0. We can define a permutation of the fiber $p^{-1}(y_0)$ by $\tilde{y}_0 \mapsto \widetilde{\bar{\eta}\cdot\gamma\cdot\eta}(1)$, where $\widetilde{\bar{\eta}\cdot\gamma\cdot\eta}$ is the lift of the loop $\bar{\eta}\cdot\gamma\cdot\eta$ beginning at $\tilde{y}_0$. If η_0 is any other path from x_0 to y_0, we have that

$\eta_0 \cdot \bar{\eta} \cdot \gamma \cdot \eta \cdot \bar{\eta}_0$ is a loop based at x_0, which is homotopic to γ (by commutativity). Hence, the action of $[\gamma]$ on $p^{-1}(y_0)$ does not depend on the choice of path and we may extend $[\gamma]$ to a deck transformation on the universal cover. Denote this map by $\Phi : \pi_1(X, x_0) \to G(\tilde{X})$.

In the other direction, given a deck transformation σ of $\tilde{X}$, we build an action by an element of $\pi_1(X, x_0)$ which agrees with σ on the fiber $p^{-1}(x_0)$. First, we choose a basepoint $\tilde{x}_0$ in the fiber. Then we have a unique homotopy class $[\tilde{\gamma}]$ of paths (relative the endpoints) from $\tilde{x}_0$ to $\sigma(\tilde{x}_0)$. We let $[\gamma]$ denote the image of this class in $\pi_1(X, x_0)$. This is the desired path.

Hence, we have constructed a map which conjugates the two actions of $\pi_1(X, x_0)$. Hence these actions are conjugate, i.e. the following diagram is commutative.

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\Phi"] \arrow[d, "\text{monodomy}"'] & G(\tilde{X}) \arrow[d, "\text{deck}"] \\
\mathrm{Aut}(p^{-1}(x_0)) \arrow[r, "="] & \mathrm{Aut}(p^{-1}(x_0))
\end{tikzcd}$$

Problem 28. Let Y be simply connected and let G act on Y via a covering space action. Therefore, we have a covering space $Y \to Y/G = X$. Let $y_0 \in Y$ and let x_0 be the projection of y_0. We define a map $\Phi : \pi_1(X, x_0) \to G$ as follows. For a homotopy class $[\gamma] \in \pi_1(X, x_0)$, there is a lift $[\tilde{\gamma}]$ to a homotopy (relative $\{0, 1\}$) class of paths in Y. The endpoint of $\tilde{\gamma}$ is gy_0 for some element $g \in G$. We let $\Phi[\gamma] = g$. This is surjective and a homomorphism by the same proof as in the locally path connected case. To see that the kernel is trivial, a loop $\tilde{\gamma}$ in Y projects to a loop γ in X, i.e. $\Phi[\gamma] = 1$. A null homotopy $\tilde{\gamma}_t$ of $\tilde{\gamma}$ projects to a null homotopy γ_t of γ, hence $\ker \Phi = 1$.

Problem 29. Let Y be path connected, locally path connected, and simply connected with covering space actions by groups $G_1, G_2 < \mathrm{Homeo}(Y)$. Suppose $\varphi : Y/G_1 \to Y/G_2$ is a homeomorphism. This map induces an isomorphism on fundamental groups and by the lifting criterion, there is a lift $\tilde{\varphi} : Y \to Y$ such that $\pi_2 \tilde{\varphi} = \varphi \pi_1$. Note that $\tilde{\varphi}$ is a homeomorphism and for $g \in G_1$, $\tilde{\varphi} g \tilde{\varphi}^{-1} \in G_2$ since

$$\pi_2 \tilde{\varphi} g \tilde{\varphi}^{-1} = \varphi \pi_1 g \tilde{\varphi}^{-1} = \varphi \pi_1 \tilde{\varphi}^{-1} = \varphi \varphi^{-1} \pi_2 = \pi_2.$$

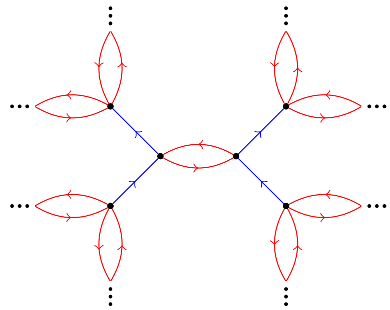
Hence, $\tilde{\varphi} G_1 \tilde{\varphi}^{-1} = G_2$.

Conversely, if G_1 and G_2 are conjugate, say $hG_1h^{-1} = G_2$, then the map $h : Y \to Y$ induces a map $\bar{h} : Y/G_1 \to Y/G_2$ given by

$$\bar{h}(G_1 y) = G_2 h(y).$$

This map is well-defined since for any $g \in G_1$, $G_2 hgy = G_2 hy$. This map has a natural inverse and so is a homeomorphism.

Problem 30. The Cayley graph of $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a, b \mid b^2 = 1 \rangle$ is shown below. The blue edges indicate the action of a and the red edges indicate the action of b.



Problem 31. Suppose $X = \mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$ (n circles) and let $\tilde{X} \to X$ be a normal cover corresponding to a normal subgroup N of F_n. We want to show that $\tilde{X}$ is the Cayley graph of $G = F_n/N$. First, fixing a basepoint $\tilde{x} \in \tilde{X}$ the vertices of $\tilde{X}$ are in one-one correspondence with the elements of G. Hence, we have a bijection Φ from the vertex set of the Cayley graph $C(G)$ to the vertex set of $\tilde{X}$ given by $\Phi(g) = g \cdot \tilde{x}$. If (v, w) is an edge in $C(G)$, this means that $w = gv$ for some generator $g \in G$. Then, $\Phi(w) = w \cdot \tilde{x} = g \cdot (v \cdot \tilde{x}) = g \cdot \Phi(v)$ and so the edge $(\Phi(v), \Phi(w))$ is in $\tilde{X}$. This gives a map $C(G) \to \tilde{X}$. The inverse map is given by taking the vertex v of $\tilde{X}$ to the vertex given by the word defined by a path from $\tilde{x}$ to v. Any two such paths γ and η define a loop $\bar{\eta} \cdot \gamma$ which then defines an element in N. Hence, we get a well-defined map $\tilde{X} \to C(G)$.

Problem 32. a) Let $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are covering spaces where $\tilde{X}_1, \tilde{X}_2$, and X are CW complexes. If restricting p_1 and p_2 to the 1-skeletons $\tilde{X}_1^1 \to X^1$ and $\tilde{X}_2^1 \to X^1$ are isomorphic via some isomorphism $\varphi : \tilde{X}_1^1 \to \tilde{X}_2^1$. Suppose φ is defined over the k-skeleton X^k. If $\phi : \partial e_k \to X^{k-1}$ is an attaching map for the CW complex X, we want to extend φ over $p_1^{-1}(e_k)$. The components of $p_1^{-1}(e_k)$ and $p_2^{-1}(e_k)$ are disjoint unions of k-cells projecting homeomorphically to e_k. If e is any such k-cell in $\tilde{X}_1$, then φ is defined over ∂e, hence determines a k-cell e' of $\tilde{X}_2$ such that $\partial e' = \varphi(\partial e)$. We can then extend φ to e via the restricted homeomorphism p_1 and the required lift of p_2. By induction, the result follows.

Conversely, if $\varphi : \tilde{X}_1 \to \tilde{X}_2$ is a covering space isomorphism, then the image of $\tilde{X}_1^1$ is $\tilde{X}_2^1$ (1-cells project homeomorphically to 1-cells under the covering maps) the restriction φ^1 to the 1-skeletons is an isomorphism since it has inverse the restriction of φ^{-1} to the 1-skeleton $\tilde{X}_2^1$.

b) Suppose $p : \tilde{X} \to X$ is a normal covering. Let $x \in X^1$ and pick any $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x)$. There is a deck transformation $\tilde{X} \to \tilde{X}$ sending $\tilde{x}_0$ to $\tilde{x}_1$. The restriction of this deck transformation to $\tilde{X}^1$ is a deck transformation of the covering space $\tilde{X}^1 \to X^1$.

Conversely, suppose by induction that $\tilde{X}^k \to X^k$ is a normal cover. Let $x \in X$ and let $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x)$. We can choose paths γ_i beginning at $\tilde{x}_i$ and ending in the boundary of the cells containing $\tilde{x}_i$ such that $p \circ \gamma_0 = p \circ \gamma_1$. There is a deck transformation over $\tilde{X}^k$ sending the two endpoints of the γ_i to one another, hence this deck transformation extends over the $(k+1)$-skeleton sending $\tilde{x}_0$ to $\tilde{x}_1$. Hence, $\tilde{X}^{k+1} \to X^{k+1}$ is a normal cover and by induction, $\tilde{X} \to X$ is a normal cover.

c) We've already shown that deck transformations of $\tilde{X}^1 \to X^1$ extend uniquely to deck transformations of $\tilde{X} \to X$. Another proof of this is as follows. Given a deck transformation σ of the 1-skeleton, suppose σ has been extended over the k-skeleton. We can extend this to the $(k+1)$-skeleton $\tilde{X}^{k+1}$. Let x be a point in the $(k+1)$-cell e. There is a path γ from x to the boundary ∂e. Then σ acts on the endpoint of γ and the path lifting property ensures that we have a new path $\tilde{\gamma}$ such that $p \circ \tilde{\gamma} = p \circ \gamma$ and with the endpoint $\sigma(\gamma(1))$. The starting point of this path, we define to be $\sigma(x)$. If η were another path to the boundary, then the composite $\bar{\eta} \cdot \gamma$ is homotopic to a path contained in ∂e relative to the endpoints. It is then possible to lift such a homotopy (by the homotopy lifting property) and so we obtain a new curve $\tilde{\eta} \cdot \tilde{\gamma}$. Hence, $\tilde{\eta}(1) = \tilde{\gamma}(1)$.

Problem 33. The subgroup K is the kernel of the action of $G_{m,n}$ on the $\mathbb{R}$ factor. The map $G_{m,n} \to \mathbb{Z}$ is given by $a \mapsto n'$ and $b \mapsto m'$. The generators a, b of the group $G_{m,n}$ acts on $T_{m,n}$ by rotating about fixed vertices v_a and v_b which are connected by an edge e. Any vertex in the $G_{m,n}$-orbit of v_a lies in the K-orbit of a vertex adjacent to v_b. This is because such a vertex has the form $a^{k_1}b^{\ell_1} \ldots a^{k_j}b^{\ell_j} \cdot v_a$. We can reduce this to the form $b^k \cdot v_a$ by multiplying by the appropriate element of K. Two such elements $b^j \cdot v_a$ and $b^k \cdot v_a$ lie in the same K-orbit iff $k \equiv j \mod n'$. Hence, there are exactly $n/n' = d$ vertices (adjacent to v_b) in the orbit of v_a. Hence, There are exactly n' distinct K-orbits of the vertex v_a. Similarly, there are m' distinct K-orbits of the vertex v_b. Also, from this we see there are d edges any two orbits of v_a and v_b.

To compute $K \cong \pi_1(T_{m,n}/K)$, which is a free group, we use the Euler characteristic. We have that $\operatorname{rank}(K) = 1 - \chi(T_{m,n}/K) = 1 - (m' + n') - dm'n'$.

Graphs and Free Groups

Problem 1. We define a metric on the graph X by specifying that each edge has length 1. The only points we need to check are the vertices. An open basis in this metric topology of a vertex v is given by the union of v with open neighborhoods of a fixed radius of the endpoints of each of the edges connecting to v. This is also a neighborhood basis in the weak topology, by definition.

The problem with an infinite number of edges incident on a single vertex is that the radii of the open balls can decrease to 0, which would not describe an open neighborhood in the metric topology.

Problem 2. Let X be a connected graph and suppose $Y \subset X$ is a connected subgraph. We define a retraction as follows. We have a homotopy equivalence from X, preserving Y, onto a graph X_0 which is Y with single edges connecting its vertices. It is easy to see here that we have a retraction given by sending the edges in $X_0 \setminus Y$ to any path connecting its vertices from within Y. We can lift this map to a retraction $X \to Y$ using the homotopy equivalence.

Problem 3. Suppose X is a tree. We show by induction on the number of vertices that $\chi(X) = 1$. If X is a vertex, then $\chi(X) = 1$. Now, suppose X is a tree and let X' be the tree obtained by deleting a terminal edge. Then, by the induction hypothesis, $\chi(X') = 1$. Thus, $\chi(X) = \chi(X') + 1 - 1 = 1$.

Given a graph X, a map which quotients out by a maximal subtree does not change the Euler characteristic. What is left is a wedge of $1 - \chi(X)$ circles. Hence, the free rank of $\pi_1(X)$ is $1 - \chi(X)$.