Root-Finding Algorithms and Hilbert's 13th Problem

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1 Introduction

Problems arising in the study of polynomials are often much easier to work with when translated into topology. Two examples are the work of Smale and McMullen on root-finding algorithms. Arnol'd's work with compositions of multivalued algebraic functions, of which Hilbert's thirteenth problem is a special case, is another.

In 1900, David Hilbert published a famous list of 23 problems which he believed would shape mathematics for the 20th century. The thirteenth problem in this list was as follows [H]:

Problem 1.1. *Consider the degree 7 polynomial*

$$z^7 + az^3 + bz^2 + cz + 1 = 0 \tag{1.1}$$

and consider z as a function in the three variables a, b, c. Is it possible to write z as a composition of a finite number of algebraic functions in two variables?

Note here that since z is multi-valued, it is to be considered as a map $\mathbb{C}^3 \to \mathrm{Sym}^7 \mathbb{C}$. The algebraic functions in the decomposition of z are also functions into $\mathrm{Sym}^d \mathbb{C}$, and by *composition* it is loosely meant any new function obtained using addition and composition of functions (this is made formal below). For example, the continuous function $(x, y) \mapsto xy$ is the composition of one-variable functions: $xy = \exp(\log(x) + \log(y))$. A variant of this problem using the composition of continuous, not algebraic, functions was proven true by Arnol'd and Kolmogorov. In fact, it was shown to hold for any three-variable continuous function. However, the algebraic version of the problem is still open.

At first glance these problems seem to be extremely disjoint and not at all connected. However, it will be seen that the solution to these problems relies heavily on translating an algebraic problem into topology and dynamics. The following table summarizes the list of problems that will be discussed:

	Algorithms used to Find Roots	Topological Problem	Topological Obstruction
Abel-Ruffini	Express a general solution in radicals	Sections over a solvable cover	Monodromy: $\pi_1(X) \to \mathfrak{S}_d$
Smale and Vassiliev	Tree traversal algorithms defined by rational maps	Covering number; Local sections	Cohomology: $H^*(\mathrm{Poly}_d^*; -)$
McMullen	Generally convergent algorithms	Attractive holomorphic families of rational maps	Monodromy: $B_n \to \mathrm{Mod}(\hat{\mathbb{C}} \setminus A)$

2 Fuchs' Cohomology Computation

In this section we give a result on the cohomologies of pure configurations and unordered configurations. This will be used as a topological obstruction to problems discussed in later sections.

Let $\mathrm{PConf}_n\mathbb{C}$ and $\mathrm{Conf}_n\mathbb{C}$ denote the ordered and unordered configuration spaces, respectively. The fundmental groups of these spaces are the pure braid group P_n and the braid group B_n, respectively.

Theorem 2.1 (Fuchs, [Fu])**.** *The ring $H^*(\mathrm{Conf}_n\mathbb{C};\mathbb{Z}_2)$ has generators $a_{m,k}$ for $m = 0, \dots, n$ and $k = 1, \dots, n$ of degree $2^k(2^m - 1)$ with relations $a_{m,k}^2 = 0$ and $a_{m_1,k_1} \cdots a_{m_s,k_s} = 0$ if and only if $2^{m_1+\cdots+m_s+k_1+\cdots+k_s} > n$.*

Proof Sketch. This theorem is proven by constructing a cellular structure on $\mathrm{Conf}_n\mathbb{C}$ where the cells of dimension k are in bijective correspondence with the ordered partitions of n into a sum of $n-k$ integers. A chain is said to be *dyadic* if each term in the partitions of n corresponding to each cell is a power of 2 and a cell is *symmetric* if it is invariant under action by $\mathfrak{S}_{n-k}$. It is then possible to realize $H^*(\mathrm{Conf}_n\mathbb{C};\mathbb{Z}_2)$ as the subring of $C^*(\mathrm{Conf}_n\mathbb{C};\mathbb{Z}_2)$ consisting of dyadic, symmetric chains. □

Fuchs uses this result to prove that the kth Stiefel-Whitney class of the vector bundle ξ_n over $\mathrm{Conf}_n\mathbb{C}$ arising from the homomorphism $B_n \to \mathfrak{S}_n \to O(n)$ is the sum of all elements in the chosen basis of $H^k(\mathrm{Conf}_n\mathbb{C};\mathbb{Z}_2)$ given in the theorem.

3 Bounds Related to Hilbert's Problem

Here, an extremely useful construction is detailed. Begin with a map $p : \mathbb{C}^k \to \mathbb{C}^n$ with polynomial coordinates. Let $f : \mathbb{C}^k \to \mathrm{Sym}^n\,\mathbb{C}$ be the *algebraic function* associated to the polynomial

$$P(x, z) = z^n + p_1(x)z^{n-1} + \cdots + p_n(x), \tag{3.1}$$

i.e. f sends $x \in \mathbb{C}^k$ to the set of roots z of polynomial $P(x, z)$. Let G_f denote the set of all $x \in \mathbb{C}^k$ such that $P(x, z) = 0$ is satisfied for exactly n distinct values of z. We assume $G_f \neq \varnothing$. Define $E_f = \{(x, z) : x \in G_f, P(x, z) = 0\}$. It is clear that the space G_f is the variety defined by $\Delta \circ p \neq 0$, where Δ is the discriminant of $P(x, z)$ with respect to z. The covering map $E_f \to G_f$ is projection onto the first coordinate. This is an n-fold cover, so that its monodromy group, denoted $\mathrm{Mon}(f)$ is a subgroup of the symmetric group $\mathfrak{S}_n$ on n elements.

Theorem 3.1 (Abel-Ruffini)**.** *The polynomial*

$$z^n + a_1z^{n-1} + \cdots + a_n = 0$$

is solvable in radicals if and only if $n \leq 4$.

The lesser known topological proof, due to Arnol'd [Zol], of this theorem is given at the end of Section 3.2.

Example 3.2. Let $d > 1$. Consider the polynomial $P(x, z) = z^d - x$. Then, $G_f = \mathbb{C} \setminus \{0\}$ and E_f consists of all pairs (x, y) where $x \neq 0$ and $y^d = x$. It is also easy to see that the monodromy group is $\mathrm{Mon}(f) \cong \mathbb{Z}/d\mathbb{Z}$.

3.1 Superpositions of Multi-valued Functions

Given two multi-valued mappings $f : X \to \mathrm{Sym}^m\,Y$ and $g : Y \to \mathrm{Sym}^n\,Z$ we can take their composition $g \circ f : X \to \mathrm{Sym}^{mn}\,Z$ by sending x to the mn points $g(y)$ where y is one of the n values of x under f. We get another mapping $g * f : X \to \mathrm{Sym}^{mn}(Y \times Z)$ by sending x to the set of points $(f(x), g(y))$, where y ranges over the values $f(x)$.

Definition 3.3. For $i = 0, \dots, N$, let $\varphi_i : \mathbb{C}^\ell \to \mathrm{Sym}^{n_i}\,\mathbb{C}$ be algebraic functions and let $p_i : \mathbb{C}^{k+i} \to \mathbb{C}^\ell$ be maps with polynomial coordinates. Let $\Phi_i = \varphi_i \circ p_i$ and define a sequence of functions inductively by $F_0 = \Phi_0 * \mathrm{id}$ and $F_i = \Phi_i * F_{i-1}$ for $i = 1, \dots, N-1$. Let $f = \Phi_N \circ F_{N-1}$. If a function can be represented in this form, it is said to be a *superposition of algebraic functions of ℓ variables.* In function notation, this gives a sequence of functions in the variable $x \in \mathbb{C}^k$:

$$F_0(x) = x,\ F_1(x) = (x, \Phi_1(x)),\ F_2(x) = \big(x, \Phi_1(x), \Phi_2(x, \Phi_1(x))\big), \dots$$

Hilbert's 13th problem, stated more formally, asks whether you can write the algebraic function defined by (1.1) as a superposition of algebraic functions of 2 variables.

Example 3.4. Let $p : \mathbb{C}^2 \to \mathbb{C}^2$ denote the map $(x_1, x_2) \mapsto (3x_1, 2x_2)$ and consider the algebraic function f sending a point $x = (x_1, x_2)$ to the roots of the polynomial $z^3 + 3x_1 z + 2x_2$. Then, defining

$$\Phi_0 = \sqrt{x_2^2 + x_1^3}, \Phi_1 = \sqrt[3]{\Phi_0 - x_2}, \Phi_2 = \sqrt[3]{-x_2 - \Phi_0},$$

it is seen that the values of $f(x)$ are a subset of the values of $(\Phi_1 + \Phi_2)(x)$. We thus represent f as an *indecomposable component* of the 18-valued map $\Phi_1 + \Phi_2$.

The superposition of algebraic functions is also algebraic. This can easily be seen by noting that the coefficients of the polynomial defining f are polynomials in the coefficients of the polynomials p_i constituting the coefficients of the algebraic functions φ_i.

3.2 Superpositions as Covering Maps

One way to think about superpositions is as a decomposition of covering spaces. Suppose f is an algebraic function represented as a superposition as in Definition 3.3. We define a sequence

$$E_f = X_N \to X_{N-1} \to \cdots \to X_0 \to G_f$$

of covering maps as follows. We let X_0 be the covering space obtained by restricting $E_{\Phi_0} \to G_{\Phi_0}$ to $G_f \subset G_{\Phi_0}$. At each step, Φ_i is a multi-valued map on $X_{i-1} \subset G_{\Phi_i}$ and we can define $X_i \to X_{i-1}$ to be the covering $E_{\Phi_i} \to G_{\Phi_i}$ restricted to X_{i-1}. This is equivalent to defining $X_i \to X_{i-1}$ inductively as the pullback of $E_{\varphi_{i-1}} \to G_{\varphi_{i-1}}$ along $p_{i-1} : X_{i-1} \to G_{\varphi_{i-1}}$ for each $i = 1, \ldots, N$.

Example 3.5. Consider the multi-valued function $f = \sqrt{z} + \sqrt[3]{z}$. It is possible to construct its associated covering space using the diagram

$$\begin{array}{ccccc} & & (\mathbb{C} \setminus \{0\})^2 & \xrightarrow{(x,y) \mapsto x+y} & \mathbb{C} \\ & & \big\downarrow {\scriptstyle (x,y) \mapsto (x^2, y^3)} & & \\ \mathbb{C} \setminus \{0\} & \xrightarrow{z \mapsto (z,z)} & (\mathbb{C} \setminus \{0\})^2 & & \end{array}$$

The pullback cover over $\mathbb{C} \setminus \{0\}$ is then the subspace $X = \{(z, x, y) : z = x^2 = y^3\} \subset (\mathbb{C} \setminus \{0\})^3$. The monodromy group $\mathrm{Mon}(f)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$.

The last example above is generalizable to the following:

Proposition 3.6. *Let f, g be algebraic functions. The monodromy group of $f + g$, fg and f/g are all quotients of the product* $\mathrm{Mon}(f) \times \mathrm{Mon}(g)$.

Proof. If the functions f and g are defined by polynomials p and q, then considering the diagram

$$\begin{array}{ccccc} X & \longrightarrow & E_f \times E_g & \xrightarrow{+} & \mathbb{C} \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Mon}(f) \times \mathrm{Mon}(g)} & & \\ G_{f+g} \subset G_f \cap G_g & \xrightarrow{z \mapsto (z,z)} & G_f \times G_g & & \end{array}$$

The branched cover $E_{f+g} \to G_{f+g}$ is an intermediate cover of $X \to G_{f+g}$. Multiplication and division are similar. ☐

Also note that composing an algebraic function f with the kth root gives a surjection $\mathrm{Mon}(\sqrt[k]{f}) \twoheadrightarrow \mathrm{Mon}(f)$ with kernel $\mathbb{Z}/k\mathbb{Z}$. Solvable groups satisfy the following properties: 1) products of solvable groups are solvable, 2) quotients of a solvable group by a normal subgroup are solvable and 3) if $G \twoheadrightarrow H$ is surjective with abelian kernel and H is solvable, then G is solvable. Thus, we find that if an algebraic function can be built from arithmetic functions and radicals, its monodromy group must be solvable. This proof of the Abel-Ruffini theorem was due to Arnol'd [Zol].

An easy consequence of this theorem is that the general cubic $z^3 + az + b$ cannot be solved without a nested radical. Suppose otherwise. Then the solution will have the form of $f(a, b) = Q(\varphi \circ P(a, b), (a, b))$, where $\varphi : \mathbb{C}^i \to \mathbb{C}^i$, for some i, is the algebraic function defined by taking some radical of each of it's components. In terms of topology, φ describes a cover $(\mathbb{C} \setminus \{0\})^i \to (\mathbb{C} \setminus \{0\})^i$ with abelian monodromy group, which contradicts the fact that the monodromy group of the algebraic function defined by $z^3 + az + b$ is $\mathfrak{S}_3$.

3.3 Bounds on Superpositions

Let $D_p(n)$ be the sum of the digits in the base p expansion of n for some prime p. The following theorem is a result due to Arnol'd [Ar]:

Theorem 3.7. *If f is the universal algebraic function given by* (3.1) *then f cannot be written as the composition of algebraic functions in $< n - D_2(n)$.*

It will turn out that if $f : \mathbb{C}^n \to \mathrm{Sym}^n \mathbb{C}$ is the universal algebraic function in n variables and if f can be represented as a superposition of algebraic functions in k variables, then it must have the form $f(x) = Q(\varphi \circ P(x), x)$, where P, Q are maps with polynomial coordinates and φ is an algebraic function in k variables. This is equivalent to saying that there are no nontrivial intermediate coverings of $E_f \to G_f$. The polynomial P defines a map $G_f \to G_\varphi$ which induces a map on cohomology with $\mathbb{Z}_2$ coefficients. If $k < n - D_2(n)$ is small, then the following facts give us the contradiction we need:

1. $H^{n-D_2(n)}(G_\varphi; \mathbb{Z}_2) = 0$ where $D_2(n)$ is the number of ones in the binary expansion of n. Proof: G_φ is a k-dimensional Stein manifold, a $2k$-dimensional Stein manifold, and is therefore homotopy equivalent to a real k-dimensional manifold.
2. The Stiefel-Whitney class $w_{n-D_2(n)}[f]$ of $E_f \to G_f$ is the image under P^* of the Stifel-Whitney class $w_{n-D_2(n)}[\varphi]$ of $E_\varphi \to G_\varphi$. Proof: the covering space $E_f \to G_f$ is isomorphic to the pullback bundle of $E_\varphi \to G_\varphi$ under the map $P : G_f \to G_\varphi$.
3. $w_{n-D_2(n)}[f] \neq 0$. Proof: this is a calculation by Fuchs [Fu].

Vassiliev [Vas] improved the lower bound to $n - D_p(n)$ for any prime p. Although this theorem cannot be applied to Hilbert's 13th, the techniques used here reduce Hilbert's 13th Problem to understanding the cohomology of the space G_f associated to (1.1).

4 Root-finding Algorithms: Smale and Vassiliev

From here on, we will let $\mathcal{P}_\epsilon(d)$ denote any algorithm with input a polynomial p and output a d-tuple of complex numbers $(\xi_1, \ldots, \xi_d)$ such that if $r_1, \ldots, r_d$ are the roots of p, then $|\xi_i - r_i| < \epsilon$ for each i. Such an algorithm is allowed to have four types of nodes:

1. Input node. We allow as input any degree d polynomial.
2. Computational node: takes the output from the previous node and enters them into a series of rational functions. The output is a tuple of numbers.
3. Branching node: takes the output from a previous computational node and compares it with zero. The algorithm then proceeds in one of two directions depending on one of the two answers.
4. Output node. This gives a d-tuple of complex numbers.

The *complexity* of such an algorithm is defined to be the number of branching nodes. Using Fuchs' [Fu] calculation, Smale [Sm] gave a lower bound of $(\log_2 d)^{2/3}$, for all $\epsilon > 0$ in some neighborhood of 0, for the complexity of $\mathcal{P}_\epsilon(d)$. This was dramatically improved by Vassiliev [Vas].

Theorem 4.1 (Vassiliev, [Vas]). *There is an $\epsilon(d) > 0$ such that the complexity of the problem $\mathcal{P}_\epsilon(d)$ is greater than or equal to $d - D_p(d)$, for all $\epsilon < \epsilon(d)$ and every prime p.*

Proof Sketch. The Schwartz genus $g(f)$ of a map $f : X \to Y$ to be the minimum number of open sets covering Y which admit a section of f. In particular, we will consider the covering map $\pi : \mathrm{PConf}_d(\mathbb{C}) \to \mathrm{Conf}_d(\mathbb{C})$, which is also a principal $\mathfrak{S}_d$-bundle. The chain of inequalities that lead up to Vassiliev's result is

$$\begin{aligned} \text{complexity of } \mathcal{P}_\epsilon(d) &\geq g(\pi) - 1 \\ &\geq h_A(\pi) - 1, \text{ for any } \mathfrak{S}_n\text{-representation } A. \end{aligned}$$

The definition of the homological genus h_A of a G-fibration $f : X \to Y$, with respect to a G-representation A is as follows. The map f induces a homotopy class of maps $c : Y \to BG$, which induces the map $c^* : H^j(BG; A) \to H^j(Y; c^*A)$ on the twisted cohomology. The homological A-genus $h_A(f)$ is defined to be the smallest i such that this map is trivial for all $j \geq i$. This reduces the proof of the theorem to calculating h_A for a certain $\mathfrak{S}_n$-representation A.

We let $A = \mathbb{Z}$ as abelian groups and let $\mathfrak{S}_n$ act via the sign representation $\mathfrak{S}_n \to \mathrm{Aut}(\mathbb{Z})$. This defines a system of twisted coefficients on $B\mathfrak{S}_n$ which can be pulled back to a system c^*A on $\mathrm{Conf}_n\mathbb{C}$ via the classifying map $c : \mathrm{Conf}_n\mathbb{C} \to B\mathfrak{S}_n$. We have the following.

Lemma 4.2. *The chain complex $C_*(\mathrm{Conf}_n\mathbb{C}; c^*A)$ is a subcomplex of $C_*(B\mathfrak{S}_n; A)$.*

Lemma 4.3. *$H^{n-1}(\mathrm{Conf}_n\mathbb{C}; c^*A)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for n a power of the prime p and 0 if n is not the power of a prime p.*

There are no cells of dimension n. Hence, $h_A(\pi) = n - 1$ if n is the power of a prime. The final bound given by Theorem 5.1 comes from partitioning $n = p^{k_1} + \cdots + p^{k_s}$ and considering a small neighborhood U of some polynomial with roots of multiplicities p^{k_i}. Restricting a fiber bundle decreases the homological genus and $U \setminus \{\Delta = 0\}$ is homeomorphic to the product $\mathrm{Conf}_{p^{k_1}}\mathbb{C} \times \cdots \times \mathrm{Conf}_{p^{k_s}}\mathbb{C}$. The result follows from the Künneth formula. □

4.1 Vassiliev's Upper Bound

In this section, we give the upper bound $n - 1 \geq$ complexity of $\mathcal{P}_\epsilon(n)$ by giving an explicit algorithm with complexity $n - 1$. This bound is also due to Vassiliev [Vas]. The basic idea is to use the Weierstraß approximation theorem (WAT) to decompose Poly_n into easy to understand subsets and then use the WAT again to approximate the roots on each of these subsets.

Let $\epsilon > 0$. First, define the metric d on the space Poly_n by

$$d(p, q) = \min_{\sigma \in \mathfrak{S}_n} \left(\max_{1 \leq i \leq n} |\xi_{\sigma(i)} - \xi'_{\sigma(i)}| \right),$$

where ξ_i are the (ordered) roots of p and ξ'_i are the roots of q.

Partition Poly_n into subsets $S_1, \ldots, S_n$ where S_t is the set of polynomials with exactly t distinct roots. For each t, let S''_t and S'_t be the $2^{-2t+1}\epsilon$-neighborhood and $2^{-2t}\epsilon$-neighborhood of S_t, respectively, so that $S_t \subset S'_t \subset S''_t$. By the WAT, there is a polynomial $\chi_t : \mathrm{Poly}_n \to \mathbb{R}$ such that

$$S'_t \subset V_t := \{\chi_t \leq 0\} \subset S''_t.$$

If $a \in V_t \setminus V_{t-1}$ then a lies in an $(2^{-2t+1}\epsilon)$-neighborhood of exactly one component of S_t. Suppose this partition corresponds to the partition $n = n_1 + \cdots + n_t$. Partition the roots of a into sets of order n_i, according to the ordering of their real parts. We can then order the real parts of these roots, then the imaginary parts (using the lexographic ordering via the sets in the partition). We obtain $2n$ polynomials $r_1, \dots, r_n, s_1, \dots, s_n$ prescribed by the WAT, approximating the mean of the roots in each set and the imaginary parts separately.

The algorithm is as follows: let a be a polynomial. Compute $\xi_1(a)$. If $\xi_1 \leq 0$, output $(r_1 + is_1, \dots, r_n + is_n)$ following the method described above. If $\xi_1(a) \geq 0$, compute ξ_2 and compare with 0 and so on. It is clear that this algorithm has complexity $n-1$.

4.2 Relation to Obstruction Theory

We now turn to a slightly different question. If $\epsilon > 0$ and d is a positive integer, let $\mathcal{P}'_\epsilon(d)$ be any algorithm that approximates a single root of a general degree d polynomial to within ϵ.

Consider the universal algebraic function f for polynomials of degree n and the covering space $\varphi : E_f \to G_f = \mathrm{Conf}_n\mathbb{C}$. Just as in Smale's [Sm] argument, complexity of $\mathcal{P}'_\epsilon(d) \geq g(\varphi) - 1$.

Theorem 4.4 (Vassiliev, [Vas])**.** *If n is the power of a prime, then $g(\varphi) = n$ and the complexity $\mathcal{P}'_\epsilon(d) \geq n-1$.*

The proof of this theorem relies on constructing the obstruction to $g(\varphi) < n$ and showing that it is nontrivial when n is the power of a prime. We define a new fibration $\Theta : X \to G_f$ by as the fiberwise join of $n-1$ copies of φ. Then, the fibers of Θ are homotopically equivalent to the wedge sum $F_n = \bigvee \mathbb{S}^{n-2}$ taken over $(n-1)^{n-1}$ copies.

Proposition 4.5 (Schwarz, [Sc])**.** *The genus $g(\varphi) < n$ if and only if Θ has a section.*

The obstruction θ to the existence of a section lies in

$$H^{n-1}(\mathrm{Conf}_n\mathbb{C}; \pi_{n-2}(F_n)) \cong H^{n-1}\left(\mathrm{Conf}_n\mathbb{C}; \tilde{\mathcal{H}}^{\otimes(n-1)}\right),$$

where $\tilde{\mathcal{H}}$ is the coefficient system corresponding to the monodromy $\pi_1(\mathrm{Conf}_n\mathbb{C}) \to \tilde{H}^0(n \text{ points})$ on reduced homology via φ. There is only a single cell e in $\mathrm{Conf}_n\mathbb{C}$ of codimension $n-1$ consisting of all configurations of points with equal real part. Order the lifts of φ over e by $\xi_1, \dots, \xi_n$. The obstruction θ is homologous to $(\xi_{n-1} - \xi_n) \otimes \cdots \otimes (\xi_1 - \xi_2)$.

We have the homomorphism $\tilde{\mathcal{H}}^{\otimes n-1} \to \bigwedge^{n-1} \tilde{\mathcal{H}}$ where the latter local system is isomorphic to the local system c^*A from above. The obstruction θ is mapped to a generator of $H^{n-1}(\mathrm{Conf}_n\mathbb{C}; c^*A)$, which is nonzero if n is the power of a prime.

5 Generally Convergent Algorithms: McMullen

The results in the following section are due to Curt McMullen [Mc].

Definition 5.1. Let Rat_k denote the space of rational maps of degree k. A *purely iterative algorithm* is a mapping $T : \mathrm{Poly}_d \to \mathrm{Rat}_k$, sending $p \in \mathrm{Poly}_d$ to $T_p \in \mathrm{Rat}_k$, such that the coefficients of T_p are rational functions in the coefficients of p. Such an algorithm is said to be *generally convergent* if there is an open set of full measure $U \subset \mathrm{Poly}_d \times \mathbb{C}$ such that $T_p^n(z)$ converges to a root of p as $n \to \infty$ for every $(p, z) \in U$.

Example 5.2. Newton's method is a basic example of a purely iterative algorithm which is generally convergent for polynomials of degree 2. This algorithm is not generally convergent for polynomials of degree 3 or more.

Additionally, we have the following map T given by

$$p(z) = z^3 + az + b \mapsto T_p(z) = z - \frac{(z^3 + az + b)(3az^2 + 9bz - a^2)}{3az^4 + 18bz^3 - 6a^2z^2 - 6abz - 9b^2 - a^3}.$$

Newton's method is generally convergent for degree 2, but not degree 3 or more. This map is obtained by applying Newton's method to the rational function

$$r(z) = \frac{p(z)}{3az^2 + 9bz - a^2}.$$

The example above is generally convergent for cubics.

The main theorem for this section is the following.

Theorem 5.3. *There does not exist a generally convergent algorithm for polynomials of degree* ≥ 4.

Let X be a complex manifold and let $T : X \to \mathrm{Rat}_k$, written $x \mapsto T_x$, be holomorphic. Let $x \in X$ be a basepoint and let $f := T_x$. The family (X, f) is said to be *attractive* if there is an open dense subset U of $\mathbb{P}^1$ of full measure and a finite set A (the *attractor*) such that $f^n(z)$ tends to a A for each z in U. We have a monodromy representation $\pi_1(X, x) \to \mathrm{Mod}(\mathbb{P}^1 \setminus A)$ and we let $\mathrm{Mod}(X, f)$ denote the image of this homomorphism. Theorem 5.3 will be derived as a result of the following theorem.

Proposition 5.4. *For any attractive family* (X, f)*, the monodromy group* $\mathrm{Mod}(X, f)$ *is finite, reducible, or fixes a point of* A*.*

For degree $d \geq 4$ polynomials, there are open sets $V \subset \mathrm{Poly}_d$ such that any purely iterative algorithm on Poly_d has monodromy group which is infinite, irreducible, and acts transitively. Theorem 5.3 follows.

Proof Sketch: Theorem 5.4. Let J be the Julia set of f and let $\mathrm{Mod}(J \cup A, f)$ be the *modular group of* f *on* $J \cup A$ defined to be the set equivalence classes of maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ which are quasiconformal, $\phi(J \cup A) = J \cup A$ and commute with f under the equivalence relation of isotopy through homeomorphisms satisfying these three conditions. The *universal monodromy group of* (A, f), denoted $\mathrm{Univ}(A, f)$, is the image of the homomorphism $\mathrm{Mod}(J \cup A, f) \to \mathrm{Mod}(\mathbb{P}^1 \setminus A)$ given by taking $[\phi]$ to its action on $\mathbb{P}^1 \setminus A$. The monodromy representation $\pi_1(X, f) \to \mathrm{Mod}(\mathbb{P}^1 \setminus A)$ factors through this homomorphism so that the monodomy group $\mathrm{Mod}(X, f)$ of an attractive family with attractor A is a subgroup of $\mathrm{Univ}(A, f)$. The final step comes by showing that if $\mathrm{Univ}(A, f)$ is irreducible and fixed-point free then it is finite. □

6 References

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