THE KERNEL OF THE MONODROMY OF THE UNIVERSAL FAMILY OF DEGREE d SMOOTH PLANE CURVES

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Abstract. We consider the parameter space $\mathcal{U}_d$ of smooth plane curves of degree d. The universal smooth plane curve of degree d is a fiber bundle $\mathcal{E}_d \to \mathcal{U}_d$ with fiber diffeomorphic to a surface Σ_g. This bundle gives rise to a monodromy homomorphism $\rho_d : \pi_1(\mathcal{U}_d) \to \mathrm{Mod}(\Sigma_g)$, where $\mathrm{Mod}(\Sigma_g) := \pi_0(\mathrm{Diff}^+(\Sigma_g))$ is the mapping class group of Σ_g. The main result of this paper is that the kernel of $\rho_4 : \pi_1(\mathcal{U}_4) \to \mathrm{Mod}(\Sigma_3)$ is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where F_∞ is a free group of countably infinite rank. In the process of proving this theorem, we show that the complement $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ of the hyperelliptic locus $\mathcal{H}_g$ in Teichmüller space $\mathrm{Teich}(\Sigma_g)$ has the homotopy type of an infinite wedge of spheres. As a corollary, we obtain that the moduli space of plane quartic curves is aspherical. The proofs use results from the Weil-Petersson geometry of Teichmüller space together with results from algebraic geometry.

1. Introduction

Let $\mathbb{P}\left(\mathrm{Sym}^d\left(\mathbb{C}^3\right)\right) = \mathbb{P}^N$, where $N = \binom{d+2}{2} - 1$, be the parameter space of plane curves of degree $d > 0$. Elements of $\mathbb{P}^N$ are homogeneous degree d polynomials in variables x, y, z. Let $\mathcal{U}_d$ denote the *parameter space of smooth* plane curves of degree d. More precisely, $\mathcal{U}_d = \mathbb{P}^N \setminus \Delta_d$ is the complement of the *discriminant locus* $\Delta_d \subset \mathbb{P}^N$ which is the set of polynomials f such that the curve $V(f) = \{p \in \mathbb{P}^2 : f(p) = 0\}$ is singular.

The *universal smooth plane curve of degree* d is the fiber bundle $\mathcal{E}_d \to \mathcal{U}_d$ defined by

$$\begin{aligned} \mathcal{E}_d := \{(f,p) \in \mathcal{U}_d \times \mathbb{P}^2 : f(p) = 0\} &\to \mathcal{U}_d \\ (f,p) &\mapsto f \end{aligned}$$

There exists a monodromy homomorphism

$$\rho_d : \pi_1\left(\mathcal{U}_d\right) \to \mathrm{Mod}(\Sigma_g),$$

where $\mathrm{Mod}(\Sigma_g) := \pi_0(\mathrm{Diff}^+(\Sigma_g))$ is the mapping class group. We omit reference to the basepoint in $\pi_1\left(\mathcal{U}_d\right)$, however, it can be taken to be the Fermat curve $f_F(x,y,z) = x^d + y^d + z^d = 0$. The homomorphism ρ_d is called the *geometric monodromy of the universal smooth plane curve of degree* d. A finite presentation for $\pi_1(\mathcal{U}_d)$ has been given by Lönne [Lö9, Main Theorem].

Two natural questions are to determine the image $\mathrm{Im}(\rho_\mathrm{d})$ and kernel $K_d := \ker(\rho_d)$. Dolgachev and Libgober have given a description of $\pi_1(\mathcal{U}_3)$ as an extension

$$0 \to \mathrm{Heis}_3(\mathbb{Z}/3\mathbb{Z}) \to \pi_1(\mathcal{U}_3) \xrightarrow{\rho_3} \mathrm{Mod}(\Sigma_1) \to 0$$

[DL81, Exact Squence 4.8] of $\mathrm{Mod}(\Sigma_1)$ by the $\mathbb{Z}/3\mathbb{Z}$-points of the 3-dimensional Heisenberg group [DL81, Page 12]

$$\mathrm{Heis}_3(\mathbb{Z}/3\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{Z}/3\mathbb{Z} \right\}$$

The action $\mathrm{Mod}(\Sigma_1) \circlearrowleft H_1(\mathrm{Heis}_3(\mathbb{Z}/3\mathbb{Z}); \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^2$ is the action on the Weierstraß points of the elliptic curve. This action is exactly the composition $\mathrm{Mod}(\Sigma_1) \xrightarrow{\Psi_1} \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, where $\Psi_1 : \mathrm{Mod}(\Sigma_1) \cong \mathrm{SL}_2(\mathbb{Z})$ is the action on $H_1(\Sigma_1; \mathbb{Z})$, see [FM12, Theorem 2.5], and $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ is the natural projection.

For higher degrees $d \geq 4$, there is an exact sequence

$$0 \to K_d \to \pi_1(\mathcal{U}_d) \xrightarrow{\rho_d} \mathrm{Mod}(\Sigma_g).$$

The map ρ_d is, in general, not surjective. However, Salter [Sal19, Theorem A] has shown that $\mathrm{Im}(\rho_{\mathrm{d}})$ always has finite index in $\mathrm{Mod}(\Sigma_g)$. For $d = 4$, Kuno has shown that $\mathrm{Im}(\rho_4) = \mathrm{Mod}(\Sigma_3)$ and that K_4 is infinite [Kun08, Proposition 6.3]. For $d = 5$, Salter [Sal16, Theorem A] shows that $\mathrm{Im}(\rho_5)$ is the stabilizer $\mathrm{Mod}(\Sigma_6)[\phi]$ of a certain spin structure ϕ on Σ_6, the spin structure $\phi = e^*\mathcal{O}(1)$ induced on Σ_6 by its embedding $e : \Sigma_6 \to \mathbb{P}^2$ as a plane curve. For odd $d \geq 5$, Salter shows that the monodromy group $\mathrm{Im}(\rho_{\mathrm{d}})$ is the stabilizer of a spin structure on Σ_g, for $g = \binom{d+1}{2}$. For even $d \geq 6$, $\mathrm{Im}(\rho_{\mathrm{d}})$ is only known to be finite index in this stabilizer, hence in $\mathrm{Mod}(\Sigma_g)$ [Sal19, Theorem A].

Another result in this vein $\pi_1(\mathcal{U}_d)$ can be found in [CT99]. Recall that $\mathrm{Mod}(\Sigma_g)$ acts on $H_1(\Sigma_g; \mathbb{Z})$ preserving the intersection form. This gives rise to the *symplectic representation* $\Psi_g : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. Consider the composition

$$\Psi_g \circ \rho_d : \pi_1(\mathcal{U}_d) \to \mathrm{Sp}_{2g}(\mathbb{Z}).$$

This representation is called the *algebraic monodromy of the universal smooth plane curve of degree d*. Carlson and Toledo show that $\tilde{K}_d := \ker(\Psi_g \circ \rho_d)$ is *large* [CT99, Theorem 1.2], i.e. there is a homomorphism $\tilde{K}_d \to G$ to a noncompact semisimple real algebraic Lie group G with Zariski-dense image.

In this paper we prove the following theorem, which is a refinement of Kuno's theorem [Kun08, Proposition 6.3] that K_4 is infinite. In the statement, $\mathrm{SMod}(\Sigma_g) < \mathrm{Mod}(\Sigma_g)$ denotes the centralizer of a fixed hyperelliptic involution, the homotopy class of an order 2 homeomorphism $\tau : \Sigma_g \to \Sigma_g$ which acts on $H_1(\Sigma_g; \mathbb{Z})$ by multiplication by -1.

Theorem 1.1. *The group K_4 is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where F_∞ is an infinite rank free group. Moreover, F_∞ has a free generating set in bijection with the set of cosets of the hyperelliptic mapping class group* $\mathrm{SMod}(\Sigma_3)$*, and*

$$H_1(K_4; \mathbb{Q}) \cong \mathbb{Q}[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$$

as $\mathrm{Mod}(\Sigma_3)$*-modules.*

The idea for the proof of Theorem 1.1 is to exhibit the cover $\mathcal{U}_4^{mark} \to \mathcal{U}_4$ corresponding to K_4 as a principal fiber bundle over the complement $\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ of the hyperelliptic locus $\mathcal{H}_3$ in Teichmüller space $\mathrm{Teich}(\Sigma_3)$. The following theorem determines the homotopy type of $\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$.

Theorem 1.2. *Let $g \geq 3$. The hyperelliptic complement* $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ *has the homotopy type of a wedge* $\bigvee_{i=1}^{\infty} S^n$ *of infinitely many n-spheres, where $n = 2g-5$.*

From Theorem 1.2, we can conclude that $\mathcal{U}_4^{mark} \to \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ is trivial and Theorem 1.1 follows.

We will also show that the structure of the group K_d is closely related to that of the hyperelliptic mapping class group. The failure of our proof method in Theorem 1.1 for degrees $d > 4$ is due to the lack of knowledge of the topology of the locus of planar curves in the moduli space of Riemann surfaces; there are many more obstructions to being planar than being hyperelliptic.

The paper is organized as follows. Section 2 recalls basic facts about the Weil-Petersson metric on Teichmüller space and the hyperelliptic locus. Section 3 introduces the geodesic length functions. These will then be used to prove Theorem 1.2. The proof of Theorem 1.1 is carried out in section 4.

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2. The Hyperelliptic Locus and the Weil-Petersson Metric

For the rest of the paper, let $g \geq 2$ unless otherwise stated. In this section we give the necessary background on Teichmüller space and its geometry. We review the Weil-Petersson metric on Teichmüller space and describe the geometric properties of the hyperelliptic locus in terms of this metric, see Proposition 2.1.

2.1. **Teichmüller Space.** We recall the basic theory of Teichmüller space and of the moduli space of Riemann surfaces of genus g. For additional background, see e.g. [FM12]. Let $\mathrm{Teich}(\Sigma_g)$ denote the Teichmüller space of genus $g \geq 2$ curves. That is, $\mathrm{Teich}(\Sigma_g)$ is the set of equivalence classes $[X, h]$ of pairs (X, h), where X is a complex curve of genus g and h is a *marking*, i.e. a homeomorphism $\Sigma_g \to X$. Two pairs (X, h) and (Y, g) are equivalent if $h \circ g^{-1} : Y \to X$ is isotopic to a biholomorphism. We will also denote such an equivalence class $[X, h]$ by $\mathcal{X}$. The (complex) dimension of $\mathrm{Teich}(\Sigma_g)$ is $3g-3$.

The mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on $\mathrm{Teich}(\Sigma_g)$ by

$$[f] \cdot [X, h] = [X, h \circ f^{-1}]$$

where $[f] \in \mathrm{Mod}(\Sigma_g)$. This action is properly discontinuous [FM12, Theorem 12.2] so that the quotient space $\mathcal{M}_g := \mathrm{Mod}(\Sigma_g) \backslash \mathrm{Teich}(\Sigma_g)$, the *moduli space of genus g Riemann surfaces*, is an orbifold. Let $\pi : \mathrm{Teich}(\Sigma_g) \to \mathcal{M}_g$ denote the quotient map. The space $\mathcal{M}_g$ can also be defined as the space of all complex curves of genus g, up to biholomorphism. Note that the orbifold fundamental group $\pi_1^{orb}(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $\mathrm{Mod}(\Sigma_g)$.

2.2. **Weil-Petersson Metric.** In this subsection we recall the Weil-Petersson (WP) metric and some of its properties. The WP metric is a certain Kähler metric on $\mathrm{Teich}(\Sigma_g)$ which gives rise to a Riemannian structure on $\mathrm{Teich}(\Sigma_g)$. For more on the Weil-Petersson metric, see the survey [Wol09].

The cotangent space $T^*_{\mathcal{X}}\mathrm{Teich}(\Sigma_g)$ at a point $\mathcal{X} = [X, h] \in \mathrm{Teich}(\Sigma_g)$ can be identified with the space $Q(X)$ of holomorphic quadratic differentials on X. Define a (co)metric on $T^*_{\mathcal{X}}\mathrm{Teich}(\Sigma_g)$ by

$$\langle\langle \varphi, \psi \rangle\rangle := \int_X \varphi\overline{\psi}(ds^2)^{-1},$$

where ds^2 is the hyperbolic metric on X and $(ds^2)^{-1}$ is its dual. The *Weil-Petersson (WP) metric* is defined to be the dual of $\langle\langle \cdot, \cdot \rangle\rangle$.

The WP metric is a $\mathrm{Mod}(\Sigma_g)$-invariant, incomplete [Wol75, Section 2], smooth Riemannian metric of negative sectional curvature [Tro86, Theorem 2]. Teichmüller space $\mathrm{Teich}(\Sigma_g)$ equipped with the WP metric is *geodesically convex* [Wol87, Subsection 5.4], meaning that any two points $\mathcal{X}, \mathcal{Y} \in \mathrm{Teich}(\Sigma_g)$ are connected by a unique geodesic. When referring to any metric properties of Teichmüller space, we will assume they are with respect to the WP metric unless otherwise stated.

2.3. **Hyperelliptic Locus.** A *hyperelliptic curve* X is a complex curve equipped with a biholomorphic involution $\tau : X \to X$ such that X/τ is isomorphic to $\mathbb{P}^1$. Such a map τ, if it exists, is called a *hyperelliptic involution*. An element $[\tau] \in \mathrm{Mod}(\Sigma_g)$ is called a *hyperelliptic mapping class* if $[\tau]^2 = 1$ and Σ_g/τ is homeomorphic to $\mathbb{P}^1$, or equivalently, if $[\tau]$ acts on $H_1(\Sigma_g; \mathbb{Z})$ by multiplication by -1.

Let $\overline{\mathcal{H}_g} \subset \mathcal{M}_g$ denote the locus of hyperelliptic curves and let $\mathcal{H}_g := \pi^{-1}(\overline{\mathcal{H}_g})$, where $\pi : \mathrm{Teich}(\Sigma_g) \to \mathcal{M}_g$ is the quotient map. The set $\mathcal{H}_g$ is called the *hyperelliptic locus*. It has (complex) dimension $2g - 1$. Note that when $g = 3$, the hyperelliptic locus $\mathcal{H}_3$ has complex codimension 1 in $\mathrm{Teich}(\Sigma_g)$.

The following proposition collects some facts that will be useful in later sections.

Proposition 2.1. *The locus $\mathcal{H}_g$ is a complex-analytic submanifold of* $\mathrm{Teich}(\Sigma_g)$*. Moreover, $\mathcal{H}_g$ has infinitely many connected components (see Figure 1). If H is any component of $\mathcal{H}_g$ then H is totally geodesic in* $\mathrm{Teich}(\Sigma_g)$ *and H is biholomorphic to* $\mathrm{Teich}(\Sigma_{0,2g+2})$*, the Teichmüller space of a sphere with $2g + 2$ punctures. In particular, each component of $\mathcal{H}_g$ is contractible.*

Proof. Let $[\tau] \in \mathrm{Mod}(\Sigma_g)$ be a hyperelliptic mapping class. Then $[\tau]$ acts on $\mathrm{Teich}(\Sigma_g)$ with fixed set

$$\mathrm{Fix}([\tau]) := \{[Y, g] \in \mathrm{Teich}(\Sigma_g) : [Y, g] = [Y, g \circ \tau]\}.$$

First, we show that

$$\mathcal{H}_g = \bigcup_{[\tau] \text{ hyperelliptic}} \mathrm{Fix}([\tau]),$$

where the union is taken over all hyperelliptic mapping classes $[\tau] \in \mathrm{Mod}(\Sigma_g)$. If $[X, h] \in \mathrm{Fix}([\tau])$ then $\tau : X \to X$ is isotopic to a biholomorphism τ_b. The map τ_b must be a hyperelliptic involution, and so

$[X,h] \in \mathcal{H}_g$. Conversely, if $[X,h] \in \mathcal{H}_g$ then there is a hyperelliptic involution $\tau : X \to X$ which is a biholomorphism and so $[X,h] \in \mathrm{Fix}([\tau])$.

If $[\tau]$ and $[\eta]$ are two distinct hyperelliptic mapping classes, then $\mathrm{Fix}([\tau]) \cap \mathrm{Fix}([\eta]) = \varnothing$. More explicitly, if $[X,h] \in \mathrm{Fix}([\tau]) \cap \mathrm{Fix}([\eta])$ then, $[\tau]$ and $[\eta]$ contain biholomorphic representatives $\tau_b, \eta_b : X \to X$. By [FK80, Section III.7.9, Corollary 2], we must have $\tau_b = \eta_b$.

Each set $\mathrm{Fix}([\tau])$ is totally geodesic in $\mathrm{Teich}(\Sigma_g)$. This follows from the uniqueness of geodesics in the WP metric: if γ is any geodesic with endpoints lying in $\mathrm{Fix}([\tau])$, then $[\tau] \cdot \gamma$ must be another geodesic with the same endpoints as γ, hence γ must be fixed by τ.

For a proof that $\mathcal{H}_g$ is a complex-analytic submanifold of $\mathrm{Teich}(\Sigma_g)$ and that each component is biholomorphic to $\mathrm{Teich}(\Sigma_{0,2g+2})$, we refer the reader to [Nag88, Section 4.1.5]. □

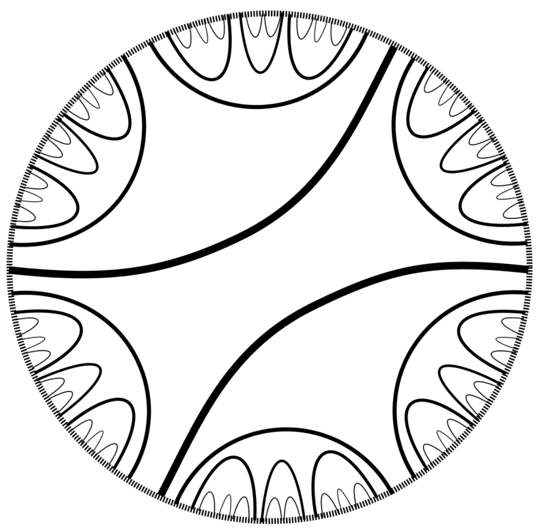

FIGURE 1. A schematic of the hyperelliptic locus $\mathcal{H}_g$ in $\mathrm{Teich}(\Sigma_g)$. The submanifold $\mathcal{H}_g \subset \mathrm{Teich}(\Sigma_g)$ has infinitely many connected components, each of which is totally geodesic with respect to the Weil-Petersson metric.

3. Homotopy Type of the Hyperelliptic Complement

In Section 3.1, we prove, Lemma 3.1, the existence of certain Morse functions on $\mathrm{Teich}(\Sigma_g)$. These functions will be used to prove Theorem 1.2 in Section 3.2.

3.1. **Geodesic Length Functions.**

This section is devoted to proving the existence of sufficiently well-behaved functions on $\mathrm{Teich}(\Sigma_g)$.

Lemma 3.1. *Let $g \geq 3$. There exists a function $f : \mathrm{Teich}(\Sigma_g) \to \mathbb{R}_+$ which satisfies the following properties.*

(1) The function f is proper, strictly convex and has positive-definite Hessian everywhere.

(2) *The function f has a unique critical point in* $\mathrm{Teich}(\Sigma_g)$*, denoted x_0.*
(3) *For any component H of $\mathcal{H}_g$, the restriction $f|_H$ has a unique critical point, denoted x_H.*
(4) *Any two critical values are distinct. That is, for any component H of $\mathcal{H}_g$, $f(x_H) \neq f(x_0)$. Also, if H' is any other component of $\mathcal{H}_g$, then $f(x_H) = f(x_{H'})$ if and only if $H = H'$.*
(5) *The set of critical values*

$$\{f(x_H) : H \text{ is a component of } \mathcal{H}_g\} \cup \{f(x_0)\}$$

is a discrete subset of $\mathbb{R}_+$.

In particular, such a function f is Morse on $\mathrm{Teich}(\Sigma_g)$ *and for each component H of $\mathcal{H}_g$, the restriction $f|_H$ is Morse.*

Proof. The function f is built using *geodesic length functions*. These functions are defined as follows. Let α be a free homotopy class of simple closed curves on Σ_g and let $[X, h]$ be a point in $\mathrm{Teich}(\Sigma_g)$. Then $h(\alpha)$ is a free homotopy class of simple closed curves in X. Recall that $h(\alpha)$ contains a unique geodesic γ. The *geodesic length function* $\ell_\alpha : \mathrm{Teich}(\Sigma_g) \to \mathbb{R}_+$ associated to α is defined by

$$\ell_\alpha(\mathcal{X}) := \text{length of the unique geodesic in the free homotopy class } h(\alpha) \text{ on } X,$$

where $\mathcal{X} = [X, h]$. Any other choice (X', h') of representative of $[X, h]$ would differ from (X, h) by an isometry, hence ℓ_α is well-defined. Fix a finite collection $\mathcal{A}$ of (homotopy classes of) simple closed curves which fills Σ_g, and let $\mathbf{c} = (c_\alpha) \in \mathbb{R}_+^{\mathcal{A}}$ be a collection of positive real numbers for each $\alpha \in \mathcal{A}$. For each choice of $\mathbf{c} \in \mathbb{R}_+^{\mathcal{A}}$, there is a function

$$\mathcal{L}_{\mathcal{A},\mathbf{c}} := \sum_{\alpha \in \mathcal{A}} c_\alpha \ell_\alpha : \mathrm{Teich}(\Sigma_g) \to \mathbb{R}_+.$$

The function f in the statement of the theorem will be defined to be $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ for a specific value of $\mathbf{c}$.

Wolpert [Wol87, Theorem 4.6] states that for any free homotopy class of simple closed curves α on Σ_g, the geodesic length function ℓ_α has positive-definite Hessian everywhere. In particular, ℓ_α is strictly convex along WP geodesics.

Recall that the Hessian operator Hess is given in local coordinates by

$$f \mapsto \left(\frac{\partial^2 f}{\partial x^i \partial x^j} + \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j,$$

where Γ_{ij}^k are the Christoffel symbols given by g. Thus, Hess is $\mathbb{R}$-linear. It follows that

$$\mathrm{Hess}\, \mathcal{L}_{\mathcal{A},\mathbf{c}} = \sum_{\alpha \in \mathcal{A}} c_\alpha \cdot (\mathrm{Hess}\, \ell_\alpha).$$

For any $v \in T_{\mathcal{X}} \mathrm{Teich}(\Sigma_g)$,

$$\mathrm{Hess}\, \mathcal{L}_{\mathcal{A},\mathbf{c}}(v, v) = \sum_{\alpha \in \mathcal{A}} c_\alpha \cdot (\mathrm{Hess}\, \ell_\alpha)(v, v) > 0$$

and so Hess $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is positive-definite. This also shows that $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is strictly convex.

Let $\mathbf{1}$ denote the element of $\mathbb{R}_+^{\mathcal{A}}$ such that $c_\alpha = 1$ for all $\alpha \in \mathcal{A}$. For $\mathbf{c} = (c_\alpha) \in \mathbb{R}_+^{\mathcal{A}}$, let $c_{min} := \min_{\alpha \in \mathcal{A}} c_\alpha$. Then,

$$c_{min}\mathcal{L}_{\mathcal{A},\mathbf{1}} \leq \mathcal{L}_{\mathcal{A},\mathbf{c}}.$$

Kerckhoff [Ker83, Lemma 3.1] states that the functions $\mathcal{L}_{\mathcal{A},\mathbf{1}}$ are proper. If $K = [a, b] \subset \mathbb{R}_+$ is compact, then

$$(\mathcal{L}_{\mathcal{A},\mathbf{c}})^{-1}(K) \subset (\mathcal{L}_{\mathcal{A},\mathbf{1}})^{-1}\left[0, b/c_{min}\right],$$

so $(\mathcal{L}_{\mathcal{A},\mathbf{c}})^{-1}(K)$ is a closed subset of a compact set, hence is compact. Thus, $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is proper. This proves (1) in the statement of the theorem.

If $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ has distinct critical points x_0 and x_0' in $\text{Teich}(\Sigma_g)$, then these are local minima of $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ since Hess $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is positive definite at both x_0 and x_0'. Without loss of generality, assume $\mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0') \leq \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0)$. However, by strict convexity, this is impossible. Let γ be the unique geodesic with $\gamma(0) = x_0$ and $\gamma(1) = x_0'$. Then

$$\mathcal{L}_{\mathcal{A},\mathbf{c}}(\gamma(t)) < (1-t)\mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0) + t\mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0') \leq \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0)$$

for all $t \in (0, 1]$, contradicting the fact that x_0 must be a local minimum. Hence $x_0 = x_0'$ and $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ has a unique critical point in $\text{Teich}(\Sigma_g)$, denoted x_0. This proves property (2).

Since the components of $\mathcal{H}_g$ are totally geodesic in the WP metric, the same argument shows that the restriction $\mathcal{L}_{\mathcal{A},\mathbf{c}}|_H$ will have a unique critical point, denoted x_H, for each component H of $\mathcal{H}_g$. This proves property (3) of the theorem. Thus, properties (1) through (3) of the theorem above are satisfied by the function $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ for any value of $\mathbf{c}$.

Let $S = \{H : H \text{ is a component of } \mathcal{H}_g\} \cup \{0\}$. For each pair $i, j \in S$ of distinct elements, there is an open dense subset $U_{i,j}$ of $\mathbb{R}_+^{\mathcal{A}}$ given by

$$U_{i,j} = \left\{\mathbf{c} \in \mathbb{R}_+^{\mathcal{A}} : \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_i) \neq \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_j)\right\}.$$

By the Baire Category Theorem, $\bigcap_{i\neq j} U_{i,j}$ is open and dense in $\mathbb{R}_+^{\mathcal{A}}$. Let $\mathbf{c}' \in \bigcap_{i\neq j} U_{i,j}$. We now define $f := \mathcal{L}_{\mathcal{A},\mathbf{c}'}$. Then, f satisfies property (4).

Lastly, we wish to show that $f(S)$ is discrete. Choose a neighborhood U_0 of x_0 and U_H of x_H, for each component H of $\mathcal{H}_g$ which are mutually disjoint. Properness of f then implies that $f(S)$ is discrete. This shows that f satisfies property (5). □

3.2. **Relative Morse theory of the pair** $(\text{Teich}(\Sigma_g), \mathcal{H}_g)$**.** The goal of this subsection is to prove Theorem 1.2. The idea is that the Morse function f found in Lemma 3.1 may be used to determine a handle decomposition of both $\mathcal{H}_g$ and $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$. For a reference on relative Morse theory, see e.g. [Sha88, Section 3].

Theorem 1.2. *Let $g \geq 3$. The hyperelliptic complement* $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ *has the homotopy type of a wedge* $\bigvee_{i=1}^{\infty} S^n$ *of infinitely many n-spheres, where $n = 2g-5$.*

Note that since every curve of genus $g = 2$ is hyperelliptic, $\mathrm{Teich}(\Sigma_2) \setminus \mathcal{H}_2 = \varnothing$. The proof of Theorem 1.2 is similar to Mess's proof that the image of the period mapping on $\mathrm{Teich}(\Sigma_2)$ has the homotopy type of an infinite wedge of circles [Mes92, Proposition 4]. We now prove Theorem 1.2.

Proof. The idea behind relative Morse theory is that such a function as given by Lemma 3.1 can be used to determine a handle decomposition not only of $\mathcal{H}_g$, but of its complement $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$. Let f be the function that satisfies the conclusion of Lemma 3.1. We let x_0 denote the unique minimum point of f in $\mathrm{Teich}(\Sigma_g)$. For each component H of $\mathcal{H}_g$, let x_H denote the unique critical point of $f|_H$. We refer to x_0 as a *critical point of f of type I* and each x_H are referred to as *critical points of f of type II*. The values $c_0 = f(x_0)$ and $c_H = f(x_H)$ are called *critical values of type I* and *II*, respectively.

For r a real number, let $X_r := \{\mathcal{X} \in \mathrm{Teich}(\Sigma_g) : f(\mathcal{X}) \leq r\}$. If $(c_0, c_0 + \epsilon]$ contains no type II critical values, then $X_{c_0+\epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to a 0-handle, i.e. a closed ball. Consider an arbitrary interval $[a, b] \subset \mathbb{R}$. If $[a, b]$ contains no critical value of type I or II of f, then $X_a \setminus \mathcal{H}_g$ is diffeomorphic to $X_b \setminus \mathcal{H}_g$. To see this, we can construct a vector field V which is equal to $\mathrm{grad}(f)$ outside a neighborhood of $\mathcal{H}_g$ and such that $V|_{\mathcal{H}_g}$ is equal to $\mathrm{grad}(f|_{\mathcal{H}_g})$. The flow along this vector field gives the required diffeomorphism.

Let x be a critical point of type II, and let $c = f(x)$. By Lemma 3.1, the set of critical values of f is discrete, so there exists some $\epsilon > 0$ such that $[c - \epsilon, c + \epsilon]$ contains no other critical values of f. We wish to show that $X_{c+\epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to $X_{c-\epsilon} \setminus \mathcal{H}_g$ with an n-handle attached, where $n = 2g - 5$ (see Figure 2).

Let H be the component of $\mathcal{H}_g$ containing x. There exists a coordinate system $(u, y) \in \mathbb{R}^{2g-4} \times \mathbb{R}^{4g-2}$ in a neighborhood U of x such that [Sha88, 3.3]

(1) $U \cap H$ is given by $u = 0$,
(2) $f = c + \|y\|^2$ on $U \cap H$.

The coordinates y are "tangent" coordinates to H and the coordinates u are "normal" coordinates to H. Note that since H has complex dimension $2g - 1$, it has real dimension $4g - 2$.

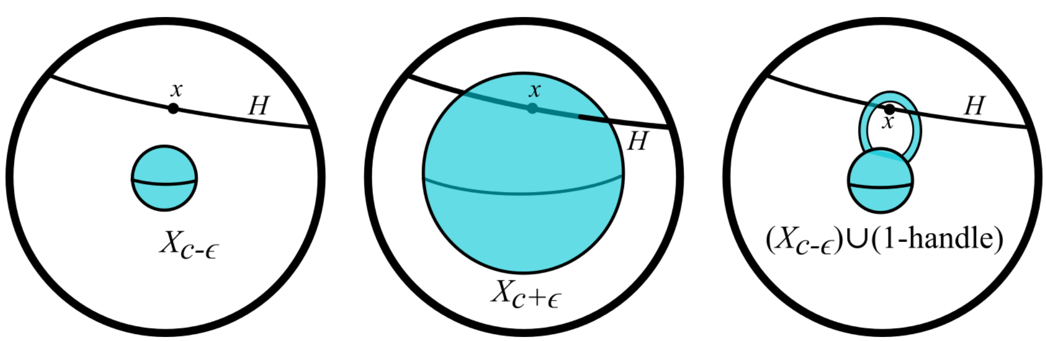

FIGURE 2. Start with $X_{c-\epsilon}$. As $c - \epsilon$ increases to $c + \epsilon$, the level set $X_{c+\epsilon}$ intersects exactly one more component H of $\mathcal{H}_g$, the component containing the critical point x.

Then, $X_{c+\epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to the union of $X_{c-\epsilon} \setminus \mathcal{H}_g$ and a tubular neighborhood of

$$\{(u,0) : \|u\|^2 = \delta\},$$

for some small $\delta > 0$. This tubular neighborhood deformation retracts to the $(2g-5)$-sphere $\{(u,0) : \|u\|^2 = \delta\}$. Hence, $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ has a handle decomposition consisting of a 0-handle with infinitely many (one for each component of $\mathcal{H}_g$) n-handles attached, where $n = 2g-5$. □

Let $\mathcal{M}_g^{nhyp}$ denote the moduli space of hyperelliptic curves of genus g. Since $\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ is a covering space for $\mathcal{M}_3^{nhyp}$, the moduli space $\mathcal{M}_3^{nhyp}$ has contractible universal cover and $\mathcal{M}_3^{nhyp}$ is aspherical. If $g \geq 4$ then $\pi_n(\mathcal{M}_g^{nhyp})$, where $n = 2g-5 > 1$, is an infinite rank abelian group. In particular, $\mathcal{M}_g^{nhyp}$ is not aspherical for $g \geq 4$.

We can be even more precise. The components of the hyperelliptic locus $\mathcal{H}_g$ are enumerated by the set of cosets of the group $\mathrm{SMod}(\Sigma_g)$ in $\mathrm{Mod}(\Sigma_g)$. Recall that $\mathrm{SMod}(\Sigma_g)$ is the centralizer in $\mathrm{Mod}(\Sigma_g)$ of a fixed hyperelliptic involution $\tau \in \mathrm{Mod}(\Sigma_g)$. The group $\mathrm{SMod}(\Sigma_g)$ is called the *hyperelliptic mapping class group of genus g*. If η is another hyperelliptic involution, then the centralizers of τ and η are conjugate in $\mathrm{Mod}(\Sigma_g)$.

Corollary 3.2. *Let $g \geq 3$. There is a homotopy equivalence*

$$\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g \cong \bigvee_{[h] \in \mathrm{Mod}(\Sigma_g)/\mathrm{SMod}(\Sigma_g)} S^{2g-5}.$$

In particular,

$$H_{2g-5}(\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g; \mathbb{Z}) \cong \mathbb{Z}[\mathrm{Mod}(\Sigma_g)/\mathrm{SMod}(\Sigma_g)]$$

as $\mathrm{Mod}(\Sigma_g)$*-modules.*

Proof. The mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on the set of components of $\mathcal{H}_g$ by permutations. Then, there is a map

$$\mathrm{Orb}(H_0) \to \mathrm{Mod}(\Sigma_g)/\mathrm{Stab}(H_0)$$
$$h \cdot H_0 \mapsto h\mathrm{Stab}(H_0)$$

from the orbit $\mathrm{Orb}(H_0)$ of H_0 to the left coset space of the stabilizer $\mathrm{Stab}(H_0)$. It suffices to show that $\mathrm{Stab}(H_0) = \mathrm{SMod}(\Sigma_g)$ and $\mathrm{Mod}(\Sigma_g)$ acts transitively on the set of components of $\mathcal{H}_g$.

First, since $H_0 = \mathrm{Fix}(\tau)$, the mapping class $h \in \mathrm{Stab}(H_0)$ if and only if

$$h \cdot \mathrm{Fix}(\tau) = \mathrm{Fix}(h\tau h^{-1}) = \mathrm{Fix}(\tau).$$

Since no hyperelliptic curve can have two distinct hyperelliptic involutions, it must follow that $h\tau h^{-1} = \tau$ so $h \in \mathrm{SMod}(\Sigma_g)$. Therefore, $\mathrm{Stab}(H_0) = \mathrm{SMod}(\Sigma_g)$.

Secondly, if H is any other component of $\mathcal{H}_g$, then $H = \mathrm{Fix}(\eta)$ for some hyperelliptic involution $\eta \in \mathrm{Mod}(\Sigma_g)$. Since hyperelliptic involutions in $\mathrm{Mod}(\Sigma_g)$ are conjugate, there exists some $h \in \mathrm{SMod}(\Sigma_g)$ such that

$$H = \mathrm{Fix}(\eta) = \mathrm{Fix}(h\tau h^{-1}) = h \cdot \mathrm{Fix}(\tau) = h \cdot H_0.$$

Therefore, $\mathrm{Mod}(\Sigma_g)$ acts transitively on the set of components of $\mathcal{H}_g$. $\square$

4. The Parameter Space of Smooth Plane Curves

In this section, we prove Proposition 4.2, showing that the cover of $\mathcal{U}_d$ determined by the subgroup K_d of $\pi_1(\mathcal{U}_d)$ carries the structure of a principal fiber bundle. This will be critical in the proof of Theorem 1.1 in Section 4.2.

4.1. Covers of $\mathcal{U}_d$ and principal fiber bundles.

The main result of this subsection is to prove Proposition 4.2, exhibiting a cover of $\mathcal{U}_d$ as a principal fiber bundle over a certain subspace of $\mathrm{Teich}(\Sigma_g)$.

Associating each point of $\mathcal{U}_d$ to the curve it determines gives rise to a map $\varphi_d : \mathcal{U}_d \to \mathcal{M}_g$ into the moduli space of Riemann surfaces of genus $g(d)$, where $g = g(d) := \binom{d-1}{2}$ by the degree-genus formula. Let $\mathcal{M}_g^{pl}$ denote the image of this map. For $d \geq 4$, the locus $\mathcal{M}_g^{pl} \subsetneq \mathcal{M}_g$ and for $d = 3$, $\mathcal{M}_1^{pl} = \mathcal{M}_1$.

There is a (disconnected) covering $\mathcal{U}_d^{mark}$ of $\mathcal{U}_d$ defined as follows. A point $(f, [h]) \in \mathcal{U}_d^{mark}$ is an ordered pair consiting of $f \in \mathcal{U}_d$ and a homotopy class $[h]$ of orientation-preserving homeomorphisms $h : \Sigma_g \to V(f)$ of some fixed Σ_g with the complex curve $V(f)$ given by $f(x, y, z) = 0$.

Let $\pi_1(\mathcal{U}_d^{mark})$ be the fundamental group of a chosen component of $\mathcal{U}_d^{mark}$. Note that $\pi_1(\mathcal{U}_d^{mark}) \cong K_d$.

Remark 4.1. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{U}_d^{mark} & \xrightarrow{\tilde{\varphi}_d} & \mathrm{Teich}(\Sigma_g) \\ \big\downarrow & & \big\downarrow \pi \\ \mathcal{U}_d & \xrightarrow[\varphi_d]{} & \mathcal{M}_g \end{array}$$

The map $\varphi_d : \mathcal{U}_d \to \mathcal{M}_g$ lifts to a map $\tilde{\varphi}_d : \mathcal{U}_d^{mark} \to \mathrm{Teich}(\Sigma_g)$ into Teichmüller space defined by

$$\varphi_d : (f, [h]) \mapsto [V(f), h].$$

Let $\mathrm{Teich}(\Sigma_g)^{pl}$ denote the image of φ_d.

Recall that a *principal G-bundle* is a fiber bundle $P \to X$ with a G-action that acts freely and transitively on the fibers.

Proposition 4.2. *For $d \geq 4$, the map $\tilde{\varphi}_d : \mathcal{U}_d^{mark} \to \mathrm{Teich}(\Sigma_g)^{pl}$ is a principal $\mathrm{PGL}_3(\mathbb{C})$-bundle.*

Proof. First, $\mathrm{PGL}_3(\mathbb{C})$ acts on $\mathcal{U}_d^{mark}$ by $g \cdot (f, [h]) = (g \cdot f, [g \circ h])$ where $g \cdot f$ denotes the action of g on polynomials $f(x, y, z)$, by acting on the triple of variables (x, y, z). This induces a map $g : V(f) \to V(g \cdot f)$ and $g \circ h$ is the composition of this map with the marking $h : \Sigma_g \to V(f)$.

This action is free. Indeed, if $g \cdot (f, [h]) = (f, [h])$ then $g \cdot f = f$ and $[g \circ h] = [h]$. Thus g induces an automorphism on the curve $V(f)$. Moreover, this automorphism acts trivially on the marking, hence trivially on $H_1(V(f); \mathbb{Z})$. An automorphism of $V(f)$ acting trivially on homology must be the identity [FM12, Theorem 6.8]. The fixed set of any automorphism of $\mathbb{P}^2$ is a linear subspace, so any $g \in \mathrm{PGL}_3(\mathbb{C})$ point-wise fixing a smooth quartic curve must be the identity automorphism.

Next, we show that this action is transitive on fibers. It suffices to show that if $\tilde{\varphi}_d(f_1, [h_1]) = \tilde{\varphi}_d(f_2, [h_2])$, then the $(f_i, [h_i])$ lie in the same $\mathrm{PGL}_3(\mathbb{C})$-orbit. By assumption, $[V(f_1), h_1] = [V(f_2), h_2]$ and there is some biholomorphism $\psi : V(f_1) \to V(f_2)$ such that $[\psi \circ h_1] = [h_2]$. Then the pullback of the hyperplane bundle H along the embeddings $e_i : V(f_i) \to \mathbb{P}^2$ gives line bundles $L_i := e_i^*(H)$ on $V(f_i)$ of degree d with $h^0(L_i) = 3$.

A g^r_d *line bundle* is a line bundle $L \to C$ such that $\deg(L) = d$ and $h^0(L) \geq r + 1$. Smooth plane curves have a unique g^2_d given by the pullback of the hyperplane bundle [Ser87, Theorem 3.13]. Therefore, L_1 and $\psi^* L_2$ are isomorphic line bundles on $V(f_1)$.

For any smooth curve C, there is a correspondence between maps $C \to \mathbb{P}^r$ up to the action of $\mathrm{PGL}_{r+1}(\mathbb{C})$ and pairs (L, V) where L is a line bundle over C and $V \subset H^0(C; L)$ is an $(r+1)$-dimensional subspace. The fact that there is a unique line bundle L on $V(f_1)$ with $h^0(L) \geq 3$ implies that there is only one such map $V(f_1) \to \mathbb{P}^2$ up to the action of $\mathrm{PGL}_3(\mathbb{C})$. Therefore, the two embeddings e_1 and $e_2 \circ \psi$ are equivalent up to the action of $\mathrm{PGL}_2(\mathbb{C})$, i.e. there is some $g \in \mathrm{PGL}_2(\mathbb{C})$ such that $g \circ e_1 = e_2 \circ \psi$. This implies that $g \cdot f_1 = f_2$ and $g : V(f_1) \to V(f_2)$ coincides with ψ. Thus, $(f_1, [h_1])$ and $(f_2, [h_2])$ lie in the same $\mathrm{PGL}_3(\mathbb{C})$-orbit.

Finally, it remains to prove local triviality. This is a consequence of a much more general fact that if G acts on a manifold P freely such that P/G is a manifold, then $q : P \to P/G$ is locally trivial. Indeed, a local trivialization of $q : P \to P/G$ can be built over any contractible subset U by first taking a section $\sigma : U \to P$ and defining $\varphi : q^{-1}(U) \to U \times G$ by $\varphi(x) = (q(x), g(x))$, where $g(x) \in G$ is the unique element such that $x = g(x) \cdot \sigma(q(x))$. □

Proposition 4.3. *Let $d \geq 3$ and $g = \binom{d-2}{2}$. The space $\mathcal{U}_d^{mark}$ has finitely many components. Consequently,* $\mathrm{Teich}(\Sigma_g)^{pl}$ *has finitely many components.*

Proof. A single component of $\mathcal{U}_d^{mark}$ is the connected covering space of $\mathcal{U}_d$ corresponding to K_d. Hence, its deck transformation group is the image of the homomorphism $\rho_d : \pi_1(\mathcal{U}_d) \to \mathrm{Mod}(\Sigma_g)$. The components of $\mathcal{U}_d^{mark}$ are enumerated by the cosets of $\mathrm{Im}(\rho_\mathrm{d})$ in $\mathrm{Mod}(\Sigma_g)$. It was shown in and [Sal19, Theorem A] that the index $[\mathrm{Mod}(\Sigma_g) : \mathrm{Im}(\rho_\mathrm{d})] < \infty$. □

4.2. **The kernel of the geometric monodromy of the universal quartic.** In this subsection, we prove Theorem 1.1.

Theorem 1.1. *The group K_4 is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where F_∞ is an infinite rank free group. Moreover, F_∞ has a free generating set in bijection with the set of cosets of the hyperelliptic mapping class group* $\mathrm{SMod}(\Sigma_3)$*, and*

$$H_1(K_4; \mathbb{Q}) \cong \mathbb{Q}[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$$

as $\mathrm{Mod}(\Sigma_3)$*-modules.*

Proof of Theorem 1.1. Classically, $\mathrm{Teich}(\Sigma_3)^{pl}$ is exactly the complement of the hyperelliptic locus $\mathcal{H}_3$ in $\mathrm{Teich}(\Sigma_3)$: the canonical map $C \to \mathbb{P}^2$ is an embedding precisely when C is nonhyperelliptic [GH94, Pages 246-7]. Consider the following principal fiber bundle.

$$\begin{array}{ccc} \mathrm{PGL}_3(\mathbb{C}) & \longrightarrow & \mathcal{U}_4^{mark} \\ & & \Big\downarrow \varphi_4 \\ & & \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3 \end{array}$$

Because $\rho_4 : \pi_1(\mathcal{U}_4) \to \mathrm{Mod}(\Sigma_3)$ is surjective [Kun08, Proposition 6.3], $\mathcal{U}_4^{mark}$ is connected.

By Theorem 1.2, $\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ is homotopy equivalent to an infinite wedge of circles and, since $\mathrm{PGL}_3(\mathbb{C})$ is connected, there must exist some continuous section $\sigma : \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3 \to \mathcal{U}_4^{mark}$. Because φ_4 is a principal $\mathrm{PGL}_3(\mathbb{C})$-bundle, the existence of such a section implies that $\mathcal{U}_4^{mark}$ is homeomorphic to $\mathrm{PGL}_3(\mathbb{C}) \times (\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3)$, and so

$$\pi_i(\mathcal{U}_4^{mark}) = \begin{cases} \mathbb{Z}/3\mathbb{Z} \times F_\infty, & \text{for } i = 1 \\ \pi_i(\mathrm{PGL}_3(\mathbb{C})), & \text{for } i > 1. \end{cases} \tag{4.1}$$

This also shows that $\pi_i(\mathcal{U}_4) \cong \pi_i(\mathrm{PGL}_3(\mathbb{C}))$ for $i \geq 2$.

We now wish to show that $H_1(K_4; \mathbb{Q})$ is isomorphic to $\mathbb{Q}\,[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$ as $\mathrm{Mod}(\Sigma_3)$-modules. The calculation of $K_4 \cong \pi_1(\mathcal{U}_4^{mark})$ in equation 4.1 shows that the projection

$$\mathcal{U}_4^{mark} \xrightarrow{\cong} \mathrm{PGL}_3(\mathbb{C}) \times (\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3) \to \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$$

induces an isomorphism

$$H_1(K_4; \mathbb{Q}) \cong H_1(\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3; \mathbb{Q}).$$

The action of $\mathrm{Mod}(\Sigma_3)$ on $\mathcal{U}_4^{mark}$ commutes with the projection map

$$\mathcal{U}_4^{mark} \to \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3,$$

so that the above isomorphism of $\mathbb{Q}$-vector spaces is an isomorphism of $\mathrm{Mod}(\Sigma_3)$-modules.

The group $H_1(\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3; \mathbb{Z})$ is the free abelian group on the set of cycles in $\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ represented by meridians around the components of the hyperelliptic locus $\mathcal{H}_3$; that is, the boundaries of disks transversely intersecting $\mathcal{H}_3$ in a single point. Such cycles are in bijection with the cosets of $\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)$ (see proof of Corollary 3.2). This bijection commutes with the action of $\mathrm{Mod}(\Sigma_3)$ and therefore this $\mathrm{Mod}(\Sigma_3)$-module is isomorphic to the permutation representation $\mathbb{Q}[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$. □

The following table shows $\pi_i(\mathcal{U}_4) \cong \pi_i(\mathrm{PGL}_3(\mathbb{C}))$ for small values of $i \geq 2$ (c.f. [MT64, Introduction], where we have used the fact that $\mathrm{SL}_3(\mathbb{C})$ covers $\mathrm{PGL}_3(\mathbb{C})$ and is homotopy equivalent to $\mathrm{SU}(3)$).

i	2	3	4	5	6	7	8	9	10	11	12
$\pi_i(\mathcal{U}_4)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	0	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/30\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/60\mathbb{Z}$

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