# Generating Extended Mapping Class Groups with Two Periodic Elements

Reid Harris

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**Abstract**

The extended mapping class group of a surface $\Sigma$ is defined to be the group of isotopy classes of (not necessarily orientation-preserving) homeomorphisms of $\Sigma$. We are able to show that the extended mapping class group of an $n$-punctured sphere is generated by two elements of finite order exactly when $n \neq 4$. We use this result to prove that the extended mapping class group of a genus 2 surface is generated by two elements of finite order.

# 1 Introduction

Let $\Sigma_{g,n}$ be an orientable, genus $g$ surface with $n$ punctures and let $\Sigma_g = \Sigma_{g,0}$. We let $\mathrm{Mod}(\Sigma_{g,n})$ denote the mapping class group of $\Sigma_{g,n}$, i.e. isotopy classes of orientation-preserving homeomorphisms $\Sigma_{g,n} \to \Sigma_{g,n}$, and let $\mathrm{Mod}^{\pm}(\Sigma_{g,n})$ be the corresponding extended mapping class group, i.e. isotopy classes of orientation-preserving or reversing homemorphisms $\Sigma_{g,n} \to \Sigma_{g,n}$. Our concern in this paper will mainly be on the groups $\mathrm{Mod}^{\pm}(\Sigma_2)$ and $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$. We consider the following question:

**Question 1.1.** *Find minimal generating sets $S$ of $\mathrm{Mod}^{\pm}(\Sigma_{g,n})$ such that each element of $S$ is of finite order.*

## 1.1 Previous Work

The problem of finding generating sets, all of whose elements satisfy a given property (e.g. finite order), is classical and has been extensively studied. In 1938, Dehn [3], proved that $\mathrm{Mod}(\Sigma_{g,0})$ was generated by $2g(g-1)$ Dehn twists for $g \geq 3$. Later, in 1964, Lickorish, [11], improved this to $g \geq 1$ and reduced the number of Dehn twists needed to $3g-1$. This was reduced further still to $2g+1$ in 1977 by Humphries, [6], using a subset of Lickorish's generating set. Johnson, [7], showed in 1983 that Humphries' generators also generate $\mathrm{Mod}(\Sigma_{g,1})$ for $g \geq 1$. Wajnryb showed in 1996 that $\mathrm{Mod}(\Sigma_{g,n})$ can be generated by two elements, however, these elements are not Dehn twists.

In regards to torsion generating sets, Maclachlan [13] showed that $\mathrm{Mod}(\Sigma_g)$ is generated by a finite set of torsion elements, concluding that moduli space is simply-connected. Luo [12] showed that $\mathrm{Mod}(\Sigma_{g,n})$ is generated by torsion elements, giving specific bounds for the order of generators given $(g, n)$. In particular, he shows that $\text{Mod}(\Sigma_{g,n})$ is generated by a involutions for $g \geq 2$. Brendle and Farb [2] show that $\text{Mod}(\Sigma_{g,n})$, for $g \geq 1$, is generated by three elements of finite order and for $g \geq 3, n = 0$ and $g \geq 4, n = 1$, $\text{Mod}(\Sigma_{g,n})$ is generated by six involutions. Kassobov [8] shows that $\text{Mod}(\Sigma_{g,n})$ can be generated by

4 involutions if $g > 7$ or $g = 7$ and $n$ is even,
5 involutions if $g > 5$ or $g = 5$ and $n$ is even,
6 involutions if $g > 3$ or $g = 3$ and $n$ is even,
9 involutions if $g = 3$ and $n$ is odd.

Korkmaz shows in [9] that $\text{Mod}(\Sigma_g)$ is generated by two elements of finite order and later showed in [10] that $\text{Mod}(\Sigma_g)$ is generated by three involutions for $g \geq 8$ and four involutions for $g \geq 3$. Yildiz [18] shows that $\text{Mod}(\Sigma_g)$ is generated by two elements of order $g$ for $g \geq 6$.

However, the corresponding question about $\text{Mod}^{\pm}(\Sigma_{g,n})$ remains largely unanswered. Du showed in [4], [17] that $\text{Mod}^{\pm}(\Sigma_1) \cong \text{GL}_2(\mathbb{Z})$ cannot be generated by two elements of finite order and, for $g > 2$, the group $\text{Mod}^{\pm}(\Sigma_g)$ is generated by two elements of finite order. Later, Altunöz et. al. in [16] showed that $\text{Mod}^{\pm}(\Sigma_g)$ is generated by three involutions for $g \geq 5$ and, moreover, $\text{Mod}^{\pm}(\Sigma_{g,n})$ can be generated by three involutions for $g = 10$, $n \geq 6$ or $g \geq 11$, $n \geq 15$. In [14], Monden shows that, for $g \geq 3$ and $n \geq 0$, the groups $\text{Mod}(\Sigma_{g,n})$ and $\text{Mod}^{\pm}(\Sigma_{g,n})$ are generated by two elements.

The question of whether $\text{Mod}^{\pm}(\Sigma_2)$ can be generated by such elements remained open. In this paper, we answer in the affirmative. In the course of the proof, we show that

**Theorem 1.2.** *The group* $\text{Mod}^{\pm}(\Sigma_{g,n})$ *can be generated by finite order elements for* $g = 0, n \neq 4$ *and* $g = 2, n = 0$. *Moreover,* $\text{Mod}^{\pm}(\Sigma_{0,4})$ *cannot be generated by finite order elements.*

# 2 Preliminaries

## 2.1 Spherical Braid Group

Given any surface $\Sigma$, the classical braid group can be generalized to the *braid group on* $\Sigma$, denoted $B_n(\Sigma) := \pi_1(\text{Conf}_n(\Sigma))$, where $\text{Conf}_n(\Sigma)$ is the space of unordered configurations of $n$ distinct points on $\Sigma$. In particular, we will be interested in the *spherical braid groups* $B_n(S^2)$. We have a surjective homomorphism $B_n \to B_n(S^2)$ with kernel generated by the central element $R_n := \sigma_1 \dots \sigma_{n-1}\sigma_{n-1} \dots \sigma_1$. Then $B_n(S^2)$ has the presentation given by generators $\tilde{\sigma}_1, \dots, \tilde{\sigma}_{n-1}$ and relations

- $\tilde{\sigma}_i\tilde{\sigma}_j = \tilde{\sigma}_j\tilde{\sigma}_i$ for $|i - j| > 2$
- $\tilde{\sigma}_i\tilde{\sigma}_j\tilde{\sigma}_i = \tilde{\sigma}_j\tilde{\sigma}_i\tilde{\sigma}_j$ for $|i - j| = 1$
- $R_n = 1$.

We turn our attention to the relationship between $B_n(S^2)$ and $\text{Mod}(\Sigma_{0,n})$. We have the exact sequence

$$0 \to \langle \beta \rangle \to B_n(S^2) \xrightarrow{\psi} \text{Mod}(\Sigma_{0,n}) \to 0 \tag{1}$$

where $\beta = (\tilde{\sigma}_1 \dots \tilde{\sigma}_{n-1})^n$ and $\langle\beta\rangle \cong \mathbb{Z}/2\mathbb{Z}$ (see [5], Section 9.1.4 and 9.2).

Here, we let $\sigma_i = \psi(\tilde{\sigma}_i)$ for $1 \le i \le n-1$. Since we are interested in elements of finite order, we record the following result:

**Proposition 2.1.** *The elements of* $\mathrm{Mod}(\Sigma_{0,n})$ *of finite order are conjugate to a power of one of the following:*

| Element | Factoring | Order |
|---|---|---|
| $\alpha_0$ | $\sigma_1 \dots \sigma_{n-1}$ | $n$ |
| $\alpha_1$ | $\sigma_1 \dots \sigma_{n-2}$ | $n-1$ |
| $\alpha_2$ | $\sigma_1 \dots \sigma_{n-3}\sigma_{n-2}^2$ | $n-2$ |

*Proof.* Let $\tilde{\sigma}_i$ refer to the standard generators of $B_n(S^2)$. Let $f \in \mathrm{Mod}(\Sigma_{0,n})$ such that $f^k = 1$. There exists a lift $\tilde{f} \in B_n(S^2)$. Thus, $\tilde{f}^k$ is a power of $\beta \in B_n(S^2)$, from (1), which has finite order and so $\tilde{f}$ is also periodic. From [15], $\tilde{f}$ must be conjugate to a power of one of

- $\tilde{\sigma}_1 \dots \tilde{\sigma}_{n-1}$,
- $\tilde{\sigma}_1 \dots \tilde{\sigma}_{n-2}\tilde{\sigma}_{n-1}^2$, or
- $\tilde{\sigma}_1 \dots \tilde{\sigma}_{n-3}\tilde{\sigma}_{n-2}^2$.

Note that $(\sigma_1 \dots \sigma_{n-2}\sigma_{n-1}^2)^{-1} = \sigma_{n-2} \dots \sigma_1$ is conjugate to $\sigma_1 \dots \sigma_{n-2}$ in $\mathrm{Mod}(\Sigma_{0,n})$. To see this, suppose $\Sigma_{0,n}$ is the unit sphere in $\mathbb{R}^3$ and arrange the marked points $p_1, \dots, p_n$ in order and uniformly along the equator of the sphere. Define $\phi : \Sigma_{0,n} \to \Sigma_{0,n}$ by rotating $\pi$ radians along the axis through $p_n$ and the center of $\Sigma_{0,n}$. Then,

$$[\phi] \cdot \sigma_i \cdot [\phi]^{-1} = \sigma_{n-1-i}$$

for all $1 \le i \le n-2$. Hence, $f$ is conjugate to a power of one of the elements in the table. □

We will also make use of the following relations, which hold in $\mathrm{Mod}(\Sigma_{0,n,0})$:

$$\alpha_0\sigma_i\alpha_0^{-1} = \sigma_{i+1} \text{ for } 1 \le i < n-1 \tag{2}$$

$$\alpha_1\sigma_i\alpha_1^{-1} = \sigma_{i+1} \text{ for } 1 \le i < n-2 \tag{3}$$

$$\alpha_2\sigma_i\alpha_2^{-1} = \sigma_{i+1} \text{ for } 1 \le i < n-3 \tag{4}$$

In particular, $\mathrm{Mod}(\Sigma_{0,n,0})$ is generated by $\sigma_1$ and $\alpha_0$.

### 2.1.1 Birman-Hilden

We introduce the Birman-Hilden exact sequence for $\Sigma_2$. For details, see [1] and [5].

**Theorem 2.2** (Birman-Hilden). *Let* $\iota \in \mathrm{Mod}(\Sigma_2)$ *denote the mapping class of an involution on* $\Sigma_2$ *with 6 fixed points. There is an exact sequence*

$$0 \to \langle\iota\rangle \to \mathrm{Mod}(\Sigma_2) \to \mathrm{Mod}(\Sigma_{0,6}) \to 0. \tag{5}$$

The following result will be useful in Section 4.3 to prove part of the main theorem. It extends the Birman-Hilden exact sequence to the extended mapping class group.

**Proposition 2.3.** *Let $\iota \in \mathrm{Mod}(\Sigma_2)$ denote the mapping class of an involution on $\Sigma_2$ with 6 fixed points. There is an exact sequence*

$$0 \to \langle \iota \rangle \to \mathrm{Mod}^{\pm}(\Sigma_2) \xrightarrow{\Psi} \mathrm{Mod}^{\pm}(\Sigma_{0,6}) \to 0.$$

*Proof.* Let $\phi \in \mathrm{Mod}^{\pm}(\Sigma_2)$ be orientation-reversing. Since there exists an orientation-reversing homeomorphism $T : \Sigma_2 \to \Sigma_2$ which is fiber-preserving, we may pick a representative $f : \Sigma_2 \to \Sigma_2$ of $\phi$ which is fiber-preserving: there is a representative $g$ of $[T]\phi$ which is fiber preserving by [1] and so we may take $f = T^{-1} \circ g$. Letting $\pi : \Sigma_2 \to \Sigma_{0,6}$ denote the branched covering map, we define $\bar{f} : \Sigma_{0,6} \to \Sigma_{0,6}$ by $\bar{f} = \pi \circ f \circ \pi^{-1}$.

Suppose $f$ and $f'$ are both representatives of $\phi$, that is, $f$ and $f'$ are isotopic. Then $T \circ f$ and $T \circ f'$ are orientation-preserving, isotopic and fiber-preserving. By Theorem 2.2, these maps are isotopic through fiber-preserving homemorphisms, say $H : \Sigma_2 \times [0,1] \to \Sigma_2$ is such an isotopy. Hence, $H' = T^{-1} \circ H$ is a fiber-preserving isotopy between $f$ and $f'$. This isotopy then descends to an isotopy between $\bar{f}$ and $\bar{f}'$. Thus, we have a well-defined map $\Psi : \mathrm{Mod}^{\pm}(\Sigma_2) \to \mathrm{Mod}^{\pm}(\Sigma_{0,6})$ given by $[f] \mapsto [\bar{f}]$. Since $\Psi|_{\mathrm{Mod}(\Sigma_2)}$ is exactly the Birman-Hilden homomorphism from (5) and the kernel of this map must lie in $\mathrm{Mod}(\Sigma_2)$, we see that $\ker(\Psi) = \langle \iota \rangle$. $\square$

# 3 Periodic Elements in $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$

Let $n \geq 1$. For our standard model of $\Sigma_{0,n}$, we take the unit sphere embedded in $\mathbb{R}^3$ along with marked points $p_k$, $k = 0, \ldots, n-1$, given by

$$p_k = \left( \cos \frac{2\pi k}{n},\ \sin \frac{2\pi k}{n},\ 0 \right).$$

Let $T : \Sigma_{0,n} \to \Sigma_{0,n}$ denote the map given by $T(x,y,z) = (x,y,-z)$. We also let $T$ denote the isotopy class of this homeomorphism in $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$. Let $\sigma_i$, for $1 \leq i \leq n-1$, denote the mapping class of the right Dehn twist about the arc connecting $p_i$ to $p_{i+1}$ along the equator. Note that $T\sigma_i = \sigma_i^{-1}T$ for each $1 \leq i \leq n-1$.

We have the following presentation for $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$: generators are $\sigma_1, \ldots, \sigma_{n-1}$, and $T$ with relations

- $T^2 = (T\sigma_i)^2 = 1$, for $1 \leq i \leq n-1$,
- $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| \geq 2$,
- $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$, for $|i-j| = 1$,
- $(\sigma_1 \ldots \sigma_{n-1})^n = 1$,
- $\sigma_1 \ldots \sigma_{n-1}\sigma_{n-1} \ldots \sigma_1 = 1$

This is the presentation obtained from the isomorphism $\mathrm{Mod}^{\pm}(\Sigma_{0,n}) \cong \mathrm{Mod}(\Sigma_{0,n}) \rtimes \mathbb{Z}/2\mathbb{Z}$ where the non-identity element $T$ of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{Mod}(\Sigma_{0,n})$ by $\sigma_i \mapsto \sigma_i^{-1}$.

Recall that the orientation-preserving mapping classes of finite order are given by Proposition 2.1. Using the presentation above, we have that

$$\begin{aligned} T\alpha_0 T &= \sigma_1^{-1} \ldots \sigma_{n-1}^{-1} \\ &= (\sigma_1 \ldots \sigma_{n-1}\sigma_{n-1} \ldots \sigma_1) \cdot \sigma_1^{-1} \ldots \sigma_{n-1}^{-1} \end{aligned}$$

$$= \sigma_1 \dots \sigma_{n-1}$$
$$= \alpha_0.$$

Thus, $T\alpha_0$ is periodic with order $n$ if $n$ is even and order $2n$ if $n$ is odd. We also easily see that

$$(T\sigma_1\sigma_3 \dots \sigma_{2k-1})^2 = 1,$$

for each $k = 0, \dots, \lfloor n/2 \rfloor$. Lastly,

$$\begin{aligned}
(T\sigma_{n-1}^{-1})\alpha_2(T\sigma_{n-1}^{-1}) &= T\sigma_{n-1}^{-1}\alpha_0\sigma_{n-1}^{-1}\sigma_{n-2}T\sigma_{n-1}^{-1} \\
&= \sigma_{n-1}\alpha_0\sigma_{n-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1} \\
&= \alpha_0\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1} \\
&= \alpha_0\sigma_{n-1}^{-1}\sigma_{n-2} \\
&= \alpha_2.
\end{aligned}$$

Thus, $T\sigma_{n-1}^{-1}$ and $\alpha_2$ commute and $T\sigma_{n-1}^{-1}\alpha_2$ has order $n-2$ if $n$ is even or $2(n-2)$ if $n$ is odd.

For general $n$, these do not exhaust all possibilities of orientation-reversion periodic elements, even up to conjugacy. For example, when $n = 9$, there exists an orientation-reversing mapping class of order 6, acting by the permutation $(1\,2\,3\,4\,5\,6)(7\,8\,9)$ on the marked points, which is not covered by any of the above examples or their powers. However, it would be interesting to find a classification of all finite-order elements of $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$ in terms of the generators $\sigma_i$.

# 4 Proof of Main Theorem

This section is divided into 3 subsections, each dealing with a proof of particular case of Theorem 1.2.

## 4.1 $\mathrm{Mod}^{\pm}(\Sigma_{0,4})$ cannot be generated by two periodic elements

**Theorem 4.1.** *The group* $\mathrm{Mod}^{\pm}(\Sigma_{0,4})$ *cannot be generated by two elements of finite order.*

*Proof.* Consider the short exact sequence

$$0 \to \langle -\mathrm{Id} \rangle \to \mathrm{GL}_2(\mathbb{Z}) \xrightarrow{q} \mathrm{PGL}_2(\mathbb{Z}) \to 0. \tag{6}$$

If $\bar{A} \in \mathrm{PGL}_2(\mathbb{Z})$ has $\bar{A}^k = \mathrm{Id} \in \mathrm{PGL}_2(\mathbb{Z})$, then for any representative $A$ of $\bar{A}$, $A^k = \pm\mathrm{Id}$ so $A$ is periodic. Suppose that $\mathrm{PGL}_2(\mathbb{Z})$ is generated by two elements $\bar{A}, \bar{B}$ of finite order. Then, if $A, B$ are representatives of $\bar{A}, \bar{B}$, then $A$ and $B$ generate a subgroup $H$ of $\mathrm{GL}_2(\mathbb{Z})$. For any $g \in \mathrm{GL}_2(\mathbb{Z})$, the only representatives of $q(g)$ are $g$ and $-g$, so either $g \in H$ or $-g \in H$. Hence, the index $[\mathrm{GL}_2(\mathbb{Z}) : H] \leq 2$. Thus, $\mathrm{GL}_2(\mathbb{Z})/H$ is abelian and $[\mathrm{GL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Z})] \leq H$. Note that $-\mathrm{Id} = [x, y]$, where

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $-\mathrm{Id} \in H$. But then $H = -H$ and so $[\mathrm{GL}_2(\mathbb{Z}) : H] = 1$ which contradicts the result from [17]. Therefore, $\mathrm{PGL}_2(\mathbb{Z})$ cannot be generated by two elements of finite order. Since we have a surjection $\mathrm{Mod}^{\pm}(\Sigma_{0,4}) \to \mathrm{PGL}_2(\mathbb{Z})$, see Section 2.2.5 of [5], the group $\mathrm{Mod}^{\pm}(\Sigma_{0,4})$ cannot be generated by two finite order elements. □

Note that $\mathrm{Mod}^{\pm}(\Sigma_{0,4})$ can be generated by the three periodic elements $T$, $T\sigma_1$, and $\alpha_0$.

## 4.2 Periodic generation of $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$, for $n \neq 4$

We begin with a simple observation:

**Proposition 4.2.** *If $n$ is odd, then $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$ is generated by $T\sigma_1$ and $T\alpha_0$.*

*Proof.* Let $H := \langle T\sigma_1, T\alpha_0 \rangle$. We have that

$$(T\alpha_0)^n = T^n \alpha_0^n = T.$$

Therefore, $T \in H$ and so $\sigma_1, \alpha_0 \in H$. Since $\sigma_1$ and $\alpha_0$ generate $\mathrm{Mod}(\Sigma_{0,n})$, we have $\mathrm{Mod}(\Sigma_{0,n}) \leq H$, but since $T \in H \setminus \mathrm{Mod}(\Sigma_{0,n})$, we must have that $H = \mathrm{Mod}^{\pm}(\Sigma_{0,n})$. □

This proposition shows that for odd $n$, the theorem is immediate since $T\sigma_1$ has order 2 and $T\alpha_0$ has order $2n$. We now turn to the more difficult case.

**Theorem 4.3.** *For all even $n \geq 6$, $\mathrm{Mod}^{\pm}(\Sigma_{0,n})$ is generated by $a = \sigma_{n-3} T \alpha_0 \sigma_{n-3}^{-1}$ and $b = T\sigma_{n-1}^{-1}\alpha_2$.*

To prove this, we proceed in a sequence of steps. Let $H = \langle a, b \rangle$. We will make use of the following relations. For $k \neq n-6, n-4, n-2$,

$$\begin{aligned} a^2 \sigma_k a^{-2} &= \sigma_{n-3}\alpha_0^2\sigma_{n-3}^{-1} \cdot \sigma_k \cdot \sigma_{n-3}\alpha_0^{-2}\sigma_{n-3}^{-1} \\ &= \sigma_{n-3}\alpha_0^2 \cdot \sigma_k \cdot \alpha_0^{-2}\sigma_{n-3}^{-1} \\ &= \sigma_{n-3}\sigma_{k+2}\sigma_{n-3}^{-1} \\ &= \sigma_{k+2}. \end{aligned}$$

**Lemma 4.4.** *We have*

$$y := \prod_{\substack{k=1 \\ k \text{ odd}}}^{n-1} \sigma_k = \sigma_1\sigma_3 \ldots \sigma_{n-1} \in H.$$

*Proof.* We first compute the following:

$$\begin{aligned} x_0 &= b^{-2}ab \\ &= \left(\alpha_2^{-2}\right) \cdot \left(\sigma_{n-3} T \alpha_0 \sigma_{n-3}^{-1}\right) \cdot \left(T\sigma_{n-1}^{-1}\alpha_2\right) \\ &= \left(\sigma_{n-2}^{-1}\sigma_{n-1}\alpha_0^{-1}\right)\left(\sigma_{n-2}^{-1}\sigma_{n-1}\alpha_0^{-1}\right) \cdot \sigma_{n-3} \boxed{T\alpha_0\sigma_{n-3}^{-1}T}\, \sigma_{n-1}^{-1}\alpha_0\sigma_{n-1}^{-1}\sigma_{n-2} \\ &= \sigma_{n-2}^{-1}\sigma_{n-1}\alpha_0^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}\alpha_0^{-1} \cdot \sigma_{n-3} \boxed{\alpha_0\sigma_{n-3}}\, \sigma_{n-1}^{-1}\alpha_0\sigma_{n-1}^{-1}\sigma_{n-2} \\ &= \sigma_{n-2}^{-1}\cancel{\sigma_{n-1}}\sigma_{n-3}^{-1}\cancel{\sigma_{n-2}}\sigma_{n-5}\sigma_{n-4}\cancel{\sigma_{n-2}^{-1}}\,\cancel{\sigma_{n-1}^{-1}}\sigma_{n-2} \end{aligned}$$

$$
= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}
$$

$$
\begin{aligned}
x_1 &= x_0 a x_0^{-1} \\
&= \left(\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\right)\cdot\sigma_{n-3}T\alpha_0\sigma_{n-3}^{-1}\cdot\left(\sigma_{n-2}^{-1}\sigma_{n-4}^{-1}\sigma_{n-5}^{-1}\sigma_{n-3}\sigma_{n-2}\right) \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\cdot\sigma_{n-3}T\alpha_0\sigma_{n-3}^{-1}\cdot\sigma_{n-2}^{-1}\sigma_{n-4}^{-1}\sigma_{n-5}^{-1}\sigma_{n-3}\sigma_{n-2} \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\sigma_{n-3}\alpha_0\sigma_{n-3}\sigma_{n-2}\sigma_{n-4}\sigma_{n-5}\sigma_{n-3}^{-1}\sigma_{n-2}^{-1}T \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}\sigma_{n-3}\sigma_{n-4}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}T\alpha_0
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= x_1 a^{-1} \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}\sigma_{n-3}\sigma_{n-4}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\boxed{T\alpha_0\cdot\sigma_{n-3}T\alpha_0^{-1}}\sigma_{n-3}^{-1} \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}\sigma_{n-3}\sigma_{n-4}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\boxed{\sigma_{n-2}^{-1}}\sigma_{n-3}^{-1} \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}\sigma_{n-1}\sigma_{n-3}\sigma_{n-4}\boxed{\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}}\sigma_{n-3}^{-1} \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}\cancel{\sigma_{n-1}}\sigma_{n-3}\sigma_{n-4}\boxed{\cancel{\sigma_{n-1}^{-1}}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}}\sigma_{n-3}^{-1} \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\boxed{\sigma_{n-3}\sigma_{n-2}\sigma_{n-3}}\sigma_{n-4}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_{n-3}^{-1} \\
&= \sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}\boxed{\sigma_{n-2}\sigma_{n-3}\cancel{\sigma_{n-2}}}\sigma_{n-4}\cancel{\sigma_{n-2}^{-1}}\sigma_{n-1}^{-1}\sigma_{n-3}^{-1} \\
&= \boxed{\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-5}}\boxed{\sigma_{n-4}\sigma_{n-2}\sigma_{n-2}}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\sigma_{n-3}^{-1} \\
&= \boxed{\sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}}\boxed{\sigma_{n-2}\sigma_{n-2}\sigma_{n-4}}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\sigma_{n-3}^{-1} \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\boxed{\sigma_{n-4}\sigma_{n-3}\sigma_{n-4}}\sigma_{n-1}^{-1}\sigma_{n-3}^{-1} \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\boxed{\sigma_{n-3}\sigma_{n-4}\cancel{\sigma_{n-3}}}\sigma_{n-1}^{-1}\cancel{\sigma_{n-3}^{-1}} \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= x_2 b^{-1} \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\cdot\sigma_{n-2}^{-1}\sigma_{n-1}\boxed{\alpha_0^{-1}\sigma_{n-1}}T \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\sigma_{n-1}\boxed{\sigma_{n-2}\alpha_0^{-1}}T \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\boxed{\sigma_{n-2}^{-1}\sigma_{n-1}\sigma_{n-2}}\alpha_0^{-1}T \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\sigma_{n-3}\sigma_{n-4}\cancel{\sigma_{n-1}^{-1}}\boxed{\cancel{\sigma_{n-1}}\sigma_{n-2}\sigma_{n-1}^{-1}}\alpha_0^{-1}T \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\sigma_{n-3}\boxed{\sigma_{n-4}\sigma_{n-2}}\sigma_{n-1}^{-1}\alpha_0^{-1}T \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\sigma_{n-2}\sigma_{n-3}\boxed{\sigma_{n-2}\sigma_{n-4}}\sigma_{n-1}^{-1}\alpha_0^{-1}T \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\boxed{\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}}\sigma_{n-4}\sigma_{n-1}^{-1}\alpha_0^{-1}T \\
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\sigma_{n-2}\boxed{\sigma_{n-3}\sigma_{n-2}\sigma_{n-3}}\sigma_{n-4}\sigma_{n-1}^{-1}\alpha_0^{-1}T
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma_{n-5}\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\boxed{\sigma_{n-2}\sigma_{n-3}\sigma_{n-2}}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\alpha_0^{-1}T \\
&= \sigma_{n-5}\cancel{\sigma_{n-2}^{-1}}\,\cancel{\sigma_{n-3}^{-1}}\boxed{\cancel{\sigma_{n-3}}\,\cancel{\sigma_{n-2}}\sigma_{n-3}}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\alpha_0^{-1}T \\
&= \sigma_{n-5}\sigma_{n-3}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\alpha_0^{-1}T
\end{aligned}
$$

$$
\begin{aligned}
x_4 &= x_3 a \\
&= \sigma_{n-5}\sigma_{n-3}\sigma_{n-3}\sigma_{n-4}\sigma_{n-1}^{-1}\boxed{\alpha_0^{-1}T\cdot\sigma_{n-3}T\alpha_0}\sigma_{n-3}^{-1} \\
&= \sigma_{n-5}\sigma_{n-3}\cancel{\sigma_{n-3}}\,\cancel{\sigma_{n-4}}\sigma_{n-1}^{-1}\boxed{\cancel{\sigma_{n-4}^{-1}}}\cancel{\sigma_{n-3}^{-1}} \\
&= \sigma_{n-5}\sigma_{n-3}\sigma_{n-1}^{-1}
\end{aligned}
$$

Define $\gamma_k := \sigma_k\sigma_{k+2}\sigma_{k+4}^{-1}$ where subscripts are taken modulo $n$. Also,

$$
\begin{aligned}
a^{2k}\gamma_1 a^{-2k} &= a^{2k}\sigma_1\sigma_3\sigma_5^{-1}a^{-2k} \\
&= a^{2k}\sigma_{2k+1}\sigma_{2k+3}\sigma_{2k+5}^{-1}a^{-2k} \\
&= \gamma_{2k+1}
\end{aligned}
$$

for all odd $k$. The above computations show that $\gamma_{n-5} \in H$. Hence, $\gamma_k \in H$ for all odd $k$. Thus,

$$
\begin{aligned}
y &= \gamma_1\gamma_3\ldots\gamma_{n-1} \\
&= \sigma_1\sigma_3\ldots\sigma_{n-3}\sigma_{n-1} \\
&\in H.
\end{aligned}
$$

One can see this by noting that each pair of the $\sigma_i$'s which appear in $y$ commute and hence, the right-hand side can be obtained by adding exponents for each $\sigma_i$ which appears. □

**Lemma 4.5.** *We have*

$$
z := \sigma_{n-2}\prod_{\substack{k=1\\ k \text{ odd}}}^{n-5}\sigma_k = \sigma_1\sigma_3\ldots\sigma_{n-5}\sigma_{n-2} \in H.
$$

*Proof.* We start with

$$
\begin{aligned}
ab &= \sigma_{n-3}T\alpha_0\sigma_{n-3}^{-1}\cdot T\sigma_{n-1}^{-1}\alpha_0\sigma_{n-1}^{-1}\sigma_{n-2} \\
&= \sigma_{n-3}\alpha_0\sigma_{n-3}\sigma_{n-1}^{-1}\alpha_0\sigma_{n-1}^{-1}\sigma_{n-2} \\
&= \left(\alpha_0\sigma_{n-1}^{-1}\right)^2\sigma_{n-5}\sigma_{n-4}\sigma_{n-2}
\end{aligned}
$$

Let $\Delta_k := \sigma_k\sigma_{k+1}\sigma_{k+3}$ for $1 \le k \le n-5$. Then,

$$
\left(\alpha_0\sigma_{n-1}^{-1}\right)^2\Delta_k = \Delta_{k+2}\left(\alpha_0\sigma_{n-1}^{-1}\right)^2
$$

for $1 \leq k \leq n-7$ and

$$\begin{aligned}
ab &= \left(\alpha_0 \sigma_{n-1}^{-1}\right)^2 \Delta_{n-5} \\
&= \alpha_0 \sigma_{n-1}^{-1} \alpha_0 \sigma_{n-1}^{-1} \sigma_{n-5} \sigma_{n-4} \sigma_{n-2} \\
&= \alpha_0^2 \sigma_{n-2}^{-1} \sigma_{n-1}^{-1} \sigma_{n-5} \sigma_{n-4} \sigma_{n-2} \\
&= \alpha_0^2 \sigma_{n-5} \sigma_{n-4} \sigma_{n-2}^{-1} \sigma_{n-1}^{-1} \sigma_{n-2} \\
&= \sigma_{n-3} \sigma_{n-2} \alpha_0^2 \sigma_{n-1} \sigma_{n-2}^{-1} \sigma_{n-1}^{-1} \\
&= \sigma_{n-3} \sigma_{n-2} \sigma_1 \left(\alpha_0 \sigma_{n-1}^{-1}\right)^2.
\end{aligned}$$

$$\begin{aligned}
\Delta_1 \Delta_3 \Delta_5 \ldots \Delta_{n-5} &= \sigma_1 \sigma_2 \sigma_4 \cdot \sigma_3 \sigma_4 \sigma_6 \cdot \sigma_5 \sigma_6 \sigma_8 \ldots \sigma_{n-7} \sigma_{n-6} \sigma_{n-4} \cdot \sigma_{n-5} \sigma_{n-4} \sigma_{n-2} \\
&= \sigma_1 \sigma_2 \sigma_3 \cdot \sigma_4 \sigma_3 \sigma_5 \cdot \sigma_6 \sigma_5 \sigma_7 \ldots \sigma_{n-6} \sigma_{n-7} \sigma_{n-5} \cdot \sigma_{n-4} \sigma_{n-5} \sigma_{n-2} \\
&= \sigma_1 \sigma_2 \ldots \sigma_{n-4} \cdot \sigma_3 \sigma_5 \ldots \sigma_{n-5} \sigma_{n-2} \\
&= \alpha_0 \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \sigma_{n-3}^{-1} \cdot \sigma_3 \sigma_5 \ldots \sigma_{n-5} \sigma_{n-2} \\
&= \alpha_0 \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \sigma_{n-3}^{-1} \cdot \sigma_1^{-1} z.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(ab)^{\frac{n}{2}-1} &= \left[\left(\alpha_0 \sigma_{n-1}^{-1}\right)^2 \Delta_{n-5}\right] \cdot \left(\alpha_0 \sigma_{n-1}^{-1}\right)^2 \Delta_{n-5} \ldots \left(\alpha_0 \sigma_{n-1}^{-1}\right)^2 \Delta_{n-5} \\
&= \left[\sigma_{n-3} \sigma_{n-2} \sigma_1 \left(\alpha_0 \sigma_{n-1}^{-1}\right)^2\right] \cdot \left(\alpha_0 \sigma_{n-1}^{-1}\right)^2 \Delta_{n-5} \ldots \left(\alpha_0 \sigma_{n-1}^{-1}\right)^2 \Delta_{n-5} \\
&= \sigma_{n-3} \sigma_{n-2} \sigma_1 \left(\alpha_0 \sigma_{n-1}^{-1}\right)^{n-2} \Delta_1 \Delta_3 \Delta_5 \ldots \Delta_{n-7} \Delta_{n-5} \\
&= \sigma_{n-3} \sigma_{n-2} \sigma_1 \left[\left(\alpha_0 \sigma_{n-1}^{-1}\right)^{n-2} \cdot \alpha_0 \sigma_{n-1}^{-1}\right] \sigma_{n-2}^{-1} \sigma_{n-3}^{-1} \cdot \sigma_1^{-1} z \\
&= \sigma_{n-3} \sigma_{n-2} \sigma_1 \sigma_{n-2}^{-1} \sigma_{n-3}^{-1} \cdot \sigma_1^{-1} z \\
&= z,
\end{aligned}$$

where we use the fact that $\alpha_0 \sigma_{n-1}^{-1} = \alpha_1$ has order $n-1$. □

*Proof of Theorem 4.3.* We have

$$\begin{aligned}
w :=& z^{-1} y \cdot \gamma_{n-3}^{-1} \\
=& \sigma_{n-2}^{-1} \sigma_{n-3} \sigma_{n-1} \cdot \sigma_{n-3}^{-1} \sigma_{n-1}^{-1} \sigma_1 \\
=& \sigma_{n-2}^{-1} \sigma_1 \\
\in& H.
\end{aligned}$$

Since

$$a^{-1} b = \sigma_{n-3} \sigma_{n-4} \sigma_{n-2}^{-1} \sigma_{n-1}^{-1} \sigma_{n-2},$$

we have that

$$c := a^{-1} b \cdot w \cdot b^{-1} a = \sigma_{n-1}^{-1} \sigma_1.$$

Thus, $T\alpha_0 \in H$ and, conjugating $\sigma_{n-3}$ by $T\alpha_0$ gives $\sigma_i \in H$ for all $1 \leq i \leq n-1$. □

## 4.3 Periodic generation of $\mathrm{Mod}^{\pm}(\Sigma_2)$

**Theorem 4.6.** *The group* $\mathrm{Mod}^{\pm}(\Sigma_2)$ *is generated by two elements of finite order.*

*Proof.* We have the exact sequence from Theorem 2.3:

$$0 \to \langle \iota \rangle \to \mathrm{Mod}^{\pm}(\Sigma_2) \xrightarrow{q} \mathrm{Mod}^{\pm}(\Sigma_{0,6}) \to 0, \tag{7}$$

where $\iota$ is the mapping class of a hyperelliptic involution, so that $\langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Let $a, b$ be as in the previous theorem and let $\tilde{a}, \tilde{b}$ be preimages to $\mathrm{Mod}^{\pm}(\Sigma_2)$. We claim that $\tilde{a}, \tilde{b}$ generate $\mathrm{Mod}^{\pm}(\Sigma_2)$. Let $H = \langle \tilde{a}, \tilde{b} \rangle$ so that $q(H) = \mathrm{Mod}^{\pm}(\Sigma_{0,6})$. For any $g \in \mathrm{Mod}^{\pm}(\Sigma_2)$, we must have either $g \in H$ or $\iota g \in H$ since these are the only two preimages of $q(g)$. Hence, $[\mathrm{Mod}^{\pm}(\Sigma_2) : H] \leq 2$.

Suppose that $[\mathrm{Mod}^{\pm}(\Sigma_2) : H] = 2$. Then the quotient map

$$\varphi : \mathrm{Mod}^{\pm}(\Sigma_2) \to \mathrm{Mod}^{\pm}(\Sigma_2)/H \cong \mathbb{Z}/2\mathbb{Z}$$

factors through the abelianization map

$$\psi : \mathrm{Mod}^{\pm}(\Sigma_2) \to (\mathbb{Z}/2\mathbb{Z})^2,$$

say $\varphi = f \circ \psi$ for some $f : (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z}$. Let $\psi' : \mathrm{Mod}^{\pm}(\Sigma_{0,6}) \to (\mathbb{Z}/2\mathbb{Z})^2$ be the abelianization of $\mathrm{Mod}^{\pm}(\Sigma_{0,6})$ given by $\psi'(\sigma_i) = (1,0)$, for $1 \leq i \leq n-1$, and $\psi'(T) = (0,1)$. Since the hyperelliptic involution is a product of 10 Dehn twists, its image in the abelianization is trivial (Section 5.1.3, [5]). Hence, $\psi = \psi' \circ q$. Since

$$\psi(\tilde{a}) = \psi'(a) = (1,1) \text{ and } \psi(\tilde{b}) = \psi'(b) = (0,1)$$

and

$$f(1,1) = \varphi(\tilde{a}) = 0 \text{ and } f(0,1) = \varphi(\tilde{b}) = 0,$$

we find that $f = 0$ and $\varphi$ is not surjective. This gives a contradiction.

□

# References


[1] J. Birman and H. Hilden. On isotopies of homeomorphisms of riemann surfaces. *The Annals of Mathematics*, 97(3):424–439, 1973.

[2] T. E. Brendle and B. Farb. Every mapping class group is generated by 6 involutions. *Journal of Algebra*, 278(1):187–198, 2004.

[3] M. Dehn. Die gruppe der abbildungsklassen. *Acta Mathematica*, 69:135–206, 1938.

[4] X. Du. The extended mapping class group can be generated by two torsions. *Journal of Knot Theory and Its Ramifications*, 26(11):1750061, 2017.

[5] X. Du. The torsion generating set of the extended mapping class groups in low genus cases. *Osaka Journal of Mathematics*, 58(4):815–825, 2021.

[6] B. Farb and D. Margalit. *A primer on mapping class groups (pms-49)*, volume 41. Princeton university press, 2011.

[7] S. P. Humphries. Generators for the mapping class group. In *Topology of Low-Dimensional Manifolds: Proceedings of the Second Sussex Conference, 1977*, pages 44–47. Springer, 2006.

[8] D. Johnson. The structure of the torelli group i: a finite set of generators for j. *Annals of Mathematics*, 118(3):423–442, 1983.

[9] M. Kassabov. Generating mapping class groups by involutions. *arXiv preprint math/0311455*, 2003.

[10] M. Korkmaz. Generating the surface mapping class group by two elements. *Transactions of the American Mathematical Society*, 357(8):3299–3310, 2005.

[11] M. Korkmaz. Mapping class group is generated by three involutions. *Mathematical Research Letters*, 27(4):1095–1108, 2020.

[12] W. B. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 60, pages 769–778. Cambridge University Press, 1964.

[13] F. Luo. Torsion elements in the mapping class group of a surface. *arXiv preprint math/0004048*, 2000.

[14] C. Maclachlan. Modulus space is simply-connected. *Proceedings of the American Mathematical Society*, 29(1):85–86, 1971.

[15] N. Monden. On minimal generating sets for the mapping class group of a punctured surface. *Israel Journal of Mathematics*, pages 1–23, 2024.

[16] K. Murasugi. Seifert fibre spaces and braid groups. *Proceedings of the London Mathematical Society*, s3-44(1):71–84, 1982.

[17] A. Tülin, M. Pamuk, and O. Yildiz. Generating the extended mapping class group by three involutions. *Osaka J. Math.*, 60(1):61–75, January 2023.

[18] O. Yıldız. Generating the mapping class group by two torsion elements. *Mediterranean Journal of Mathematics*, 19(2):59, 2022.