MATH 15200

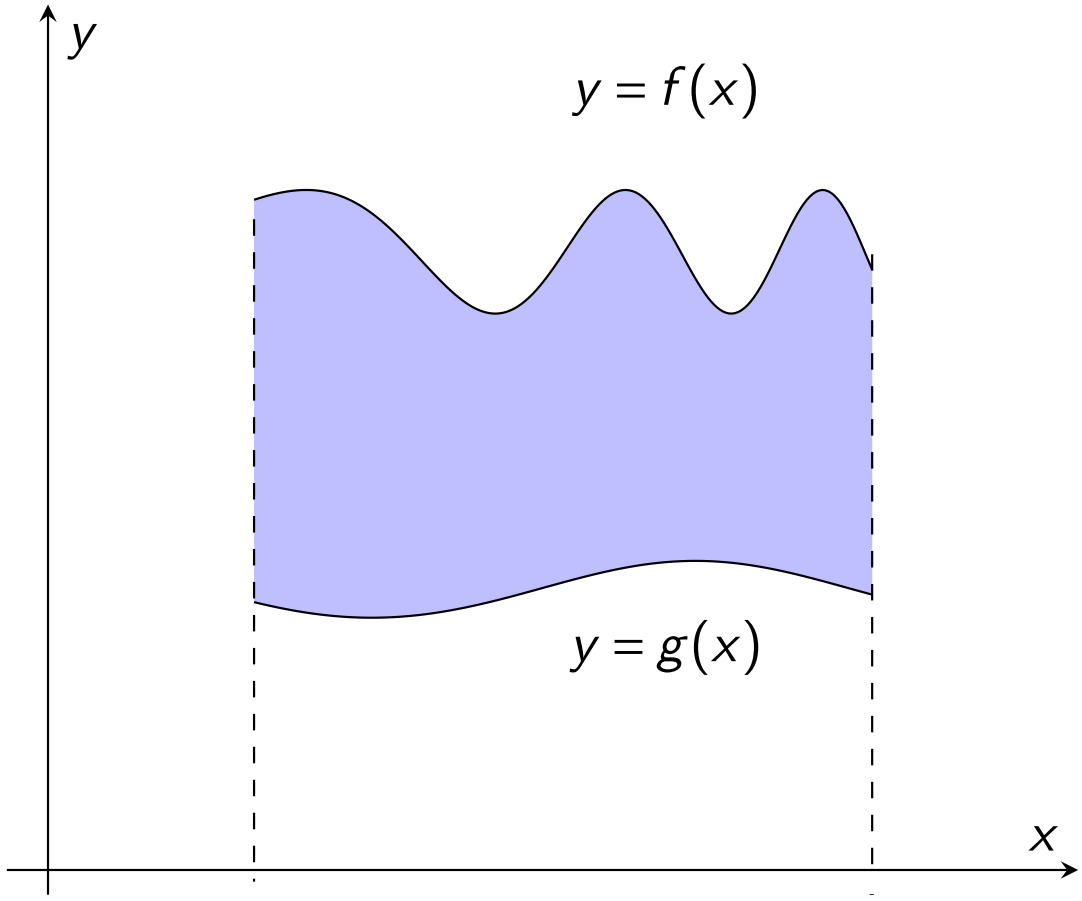
Calculus

University of Chicago

February 7, 2020

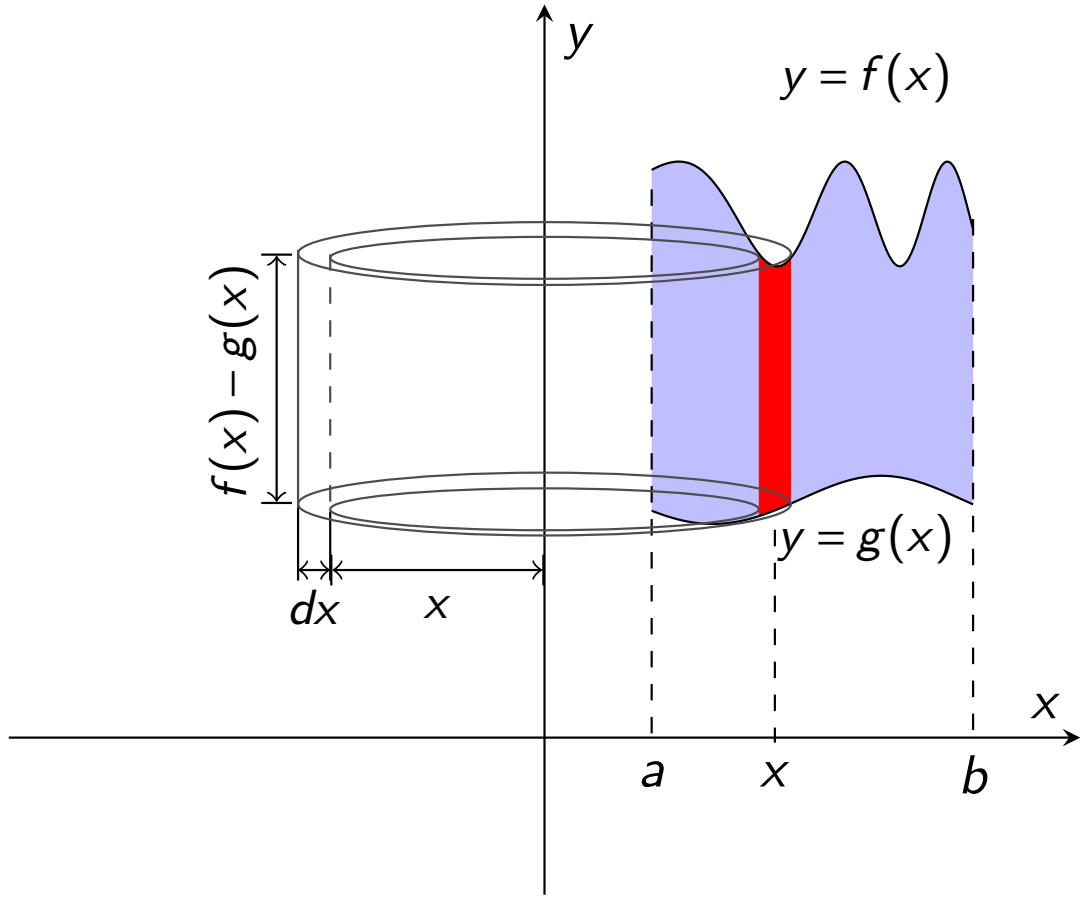
Section 6.3: Shell Method

This is another method for calculating the volumes of solids of revolution. The idea is as follows: suppose I have the following region.



I want to calculate the volume of the solid obtained by revolving this region about the y-**axis**.

Take a thin vertical slice (at x, of thickness dx) of the region and revolve this strip around the y-axis.



We obtain a shell of height $f(x)-g(x)$, thickness dx, and inner radius x.

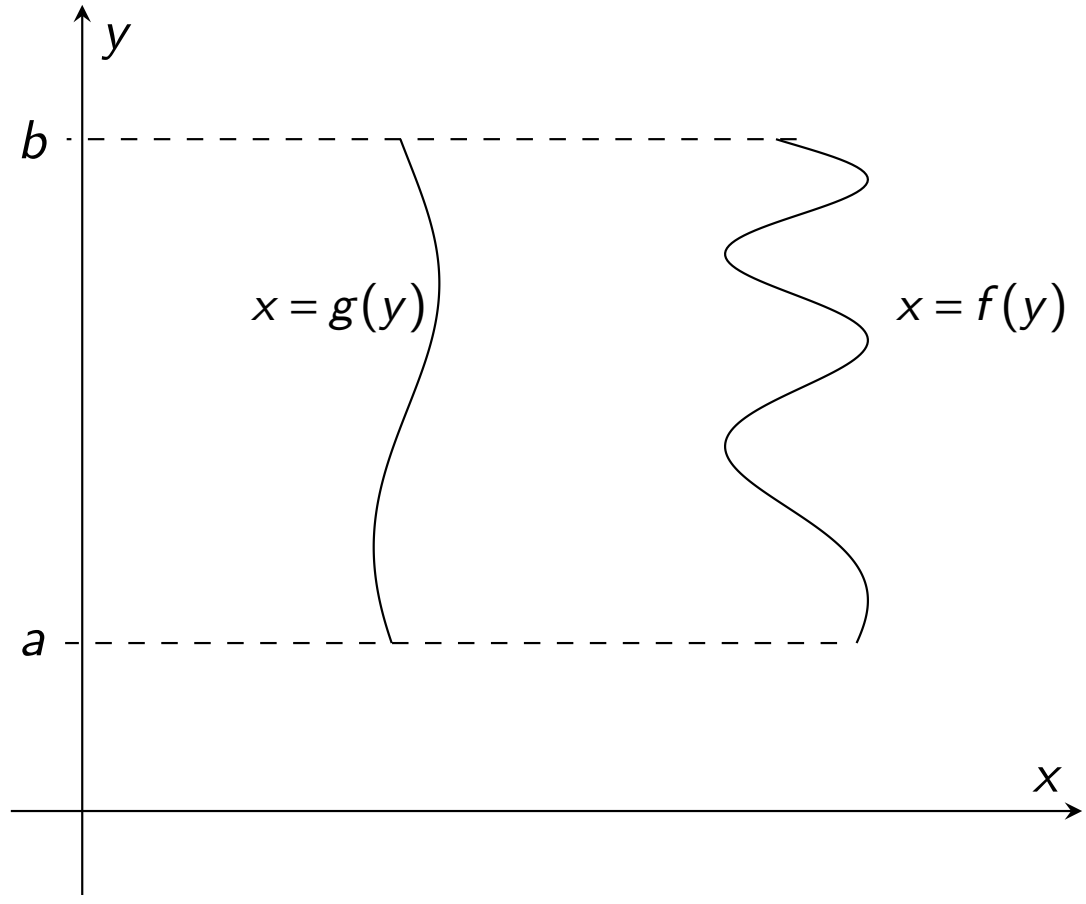
Shell Method

The volume dV of such a shell is the area of the base times the height $f(x) - g(x)$.
The base is an annulus, so the area is

$$\begin{aligned}
&\pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right) \\
&= \pi\left((x+dx)^2 - x^2\right) \\
&= \pi\left(x^2 + 2x\,dx + dx^2 - x^2\right) \\
&= \pi\left(2x\,dx + dx^2\right) \\
&\approx 2\pi x\,dx.
\end{aligned}$$

Since dx is small, dx^2 is really, really small and is negligible.
The volume of a single shell is $dV = 2\pi x(f(x) - g(x))\,dx$.
We add up all of these volumes by integrating from a to b:

$$\text{Vol}(S) = 2\pi \int_a^b x(f(x) - g(x))\,dx.$$



Rotating this region about the x-axis gives a solid S whose volume is

$$\mathrm{Vol}(S) = 2\pi \int_a^b y(f(y) - g(y))\, dy.$$

Example

Find the volume of the solid obtained by revolving the region bounded by

$$y = (x-1)^2$$

and

$$y = 1$$

about the y-axis.

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$$\begin{aligned}\int_0^1 2\pi x(1-(x-1)^2)\,dx &= 2\pi\int_0^1 (2x^2 - x^3)\,dx \\ &= 2\pi\left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)_0^1 \\ &= \frac{5\pi}{6}.\end{aligned}$$

Which method should I use (Washer or Shell)?

Very rustic guide:

Question 1: Are you revolving around the x-axis or the y-axis?

Question 2: If you wanted to calculate the area of the figure, which method would you use? I.e. would you integrate with respect to x or y?

	x-axis	y-axis
Integrate with x	Washer	Shell
Integrate with y	Shell	Washer

Example

Find the volume of the solid obtained by revolving the region bounded by

$$x^2 = 4y$$

and

$$y = \frac{1}{2}x$$

about the y-axis.

What if we revolved around the x-axis?

Example

Find the volume of the solid obtained by revolving the region bounded by

$$y = \sin(x^2)$$

with

$$0 \leq x \leq \sqrt{\pi}$$

about the y-axis.

Additional Problems:

- If you are revolving around an axis A other than the x-axis or y-axis, you need to apply some transformation that moves A to the one of these axes. If A is horizontal or vertical, this typically amounts to translating.

- If you are revolving around an axis A that passes through the region, then you have to account for *double counting*. That is, two different pieces of the region may sweep out the same piece of the solid.