MATH 15200

Calculus

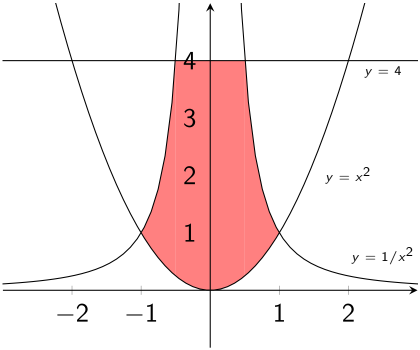
University of Chicago

January 29, 2020

Section 6.1: Area

Example

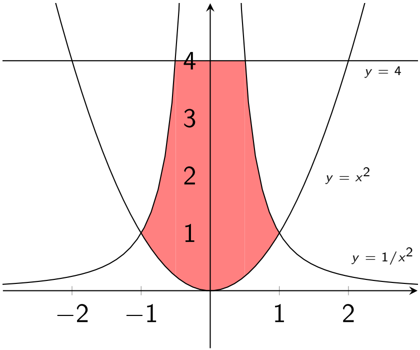
Find the area of the following figure:



Section 6.1: Area

Example

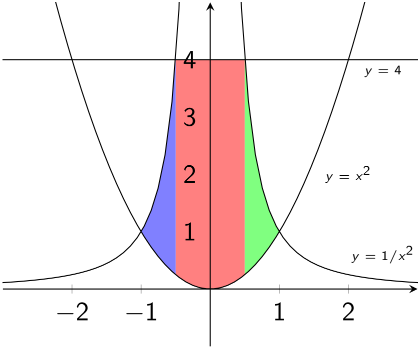
Step 1: Break up the region using vertical and horizontal lines.



Section 6.1: Area

Example

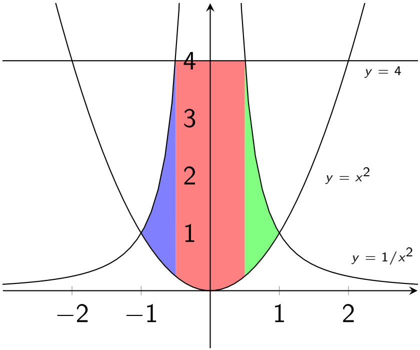
Step 1: Break up the region using vertical and horizontal lines. We get 3 distinct regions.



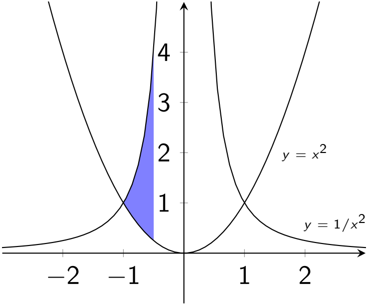
Section 6.1: Area

Example

Step 2: Calculate the area of each region and add the results.



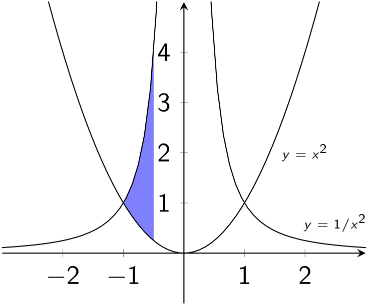
Example: Blue Region



For the blue region, we have

$$\text{Area} = \int_{-1}^{-1/2} \left(\frac{1}{x^2} - x^2 \right) dx$$

Example: Blue Region

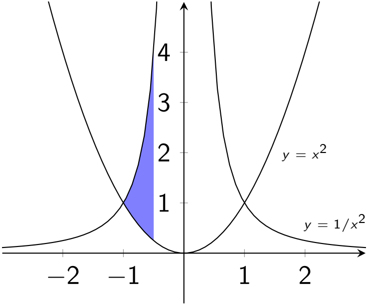


For the blue region, we have

$$\text{Area} = \int_{-1}^{-1/2} \left(\frac{1}{x^2} - x^2 \right) \, dx$$

$$= \left(-x^{-1} - \frac{1}{3}x^3 \right)_{-1}^{-1/2}$$

Example: Blue Region



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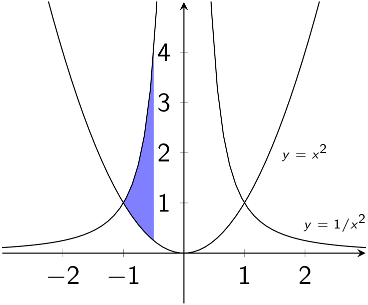

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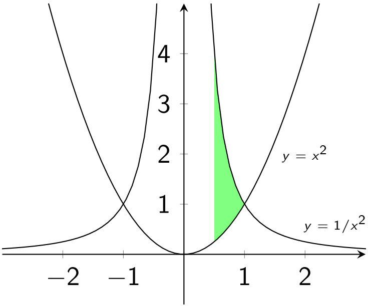
&= \left(2 + \frac{1}{24} \right) - \left(1 + \frac{1}{3} \right) \\
&= \frac{17}{24}
\end{aligned}$$

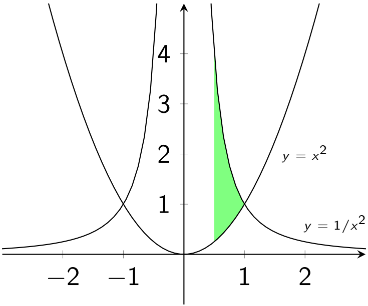
Example: Green Region



For the green region, we have

$$\text{Area} = \int_{1/2}^{1} \left(\frac{1}{x^2} - x^2 \right) \, dx$$

Example: Green Region



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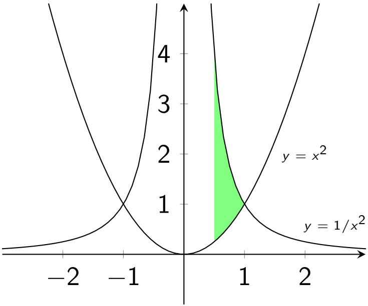

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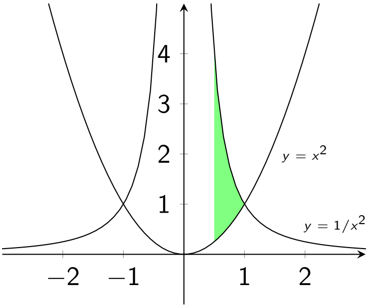
Example: Green Region



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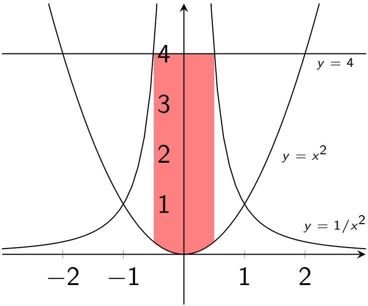
$$= \frac{17}{24}$$

Example: Red Region



For the red region, we have

$$\text{Area} = \int_{-1/2}^{1/2} \left(4 - x^2\right)\, dx$$

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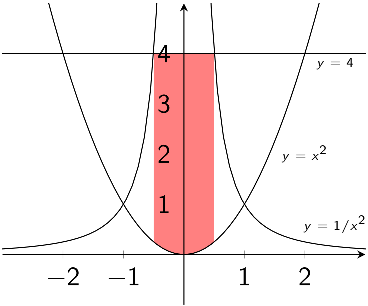

For the red region, we have

$$\text{Area} = \int_{-1/2}^{1/2} \left(4 - x^2\right)\, dx$$

$$= \left(4x - \frac{1}{3}x^3\right)_{-1/2}^{1/2}$$

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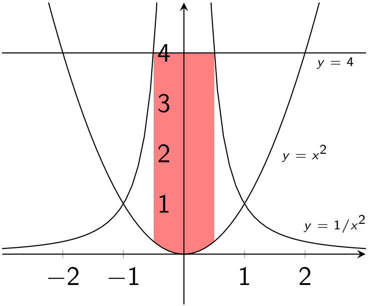

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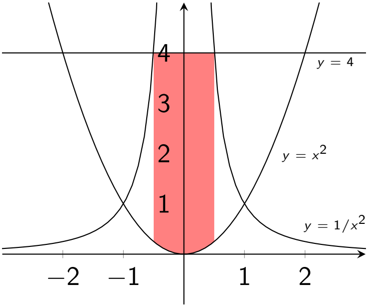
Example: Red Region



For the red region, we have

$$\begin{aligned}\text{Area} &= \int_{-1/2}^{1/2} \left(4 - x^2\right)\, dx \\ &= \left(4x - \frac{1}{3}x^3\right)_{-1/2}^{1/2} \\ &= \left(2 - \frac{1}{24}\right) - \left(-2 + \frac{1}{24}\right) \\ &= \frac{47}{12}\end{aligned}$$

Example: Conclusion

Step 3: Therefore, the total area is:

$$A_{\text{blue}} + A_{\text{green}} + A_{\text{red}} = \frac{17}{24} + \frac{47}{12} + \frac{17}{24}$$

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\begin{aligned}
A_{\text{blue}} + A_{\text{green}} + A_{\text{red}} &= \frac{17}{24} + \frac{47}{12} + \frac{17}{24} \\
&= \frac{17}{24} + \frac{94}{24} + \frac{17}{24}
\end{aligned}
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&= \frac{17}{24} + \frac{94}{24} + \frac{17}{24} \\
&= \frac{128}{24}
\end{aligned}
$$

Example: Conclusion

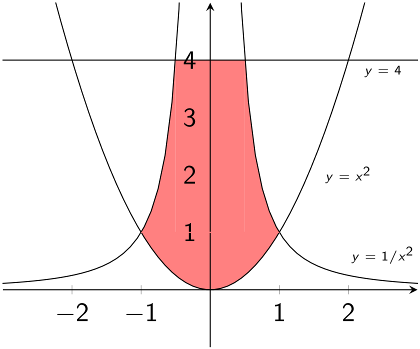
Step 3: Therefore, the total area is:

$$\begin{aligned} A_{\text{blue}} + A_{\text{green}} + A_{\text{red}} &= \frac{17}{24} + \frac{47}{12} + \frac{17}{24} \\ &= \frac{17}{24} + \frac{94}{24} + \frac{17}{24} \\ &= \frac{128}{24} \\ &= \frac{16}{3} \end{aligned}$$

Example: Another method

In this example, we integrated with respect to x.

If we integrated with respect to y, we would break up our region using horizontal lines as follows:



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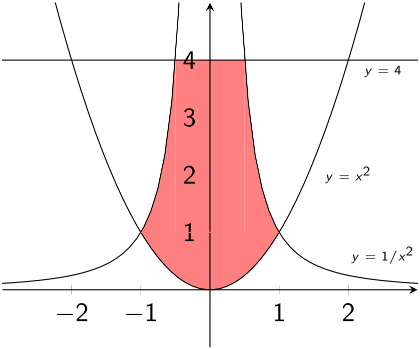

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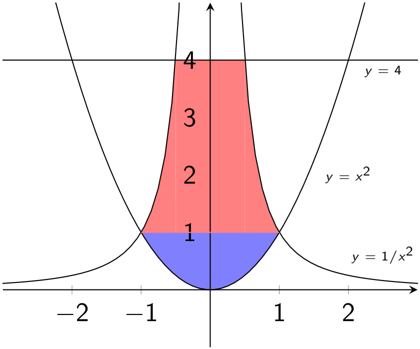
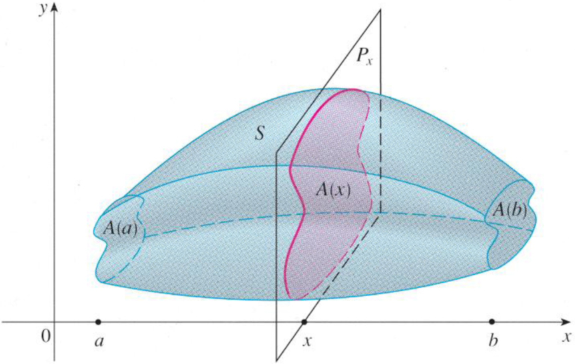


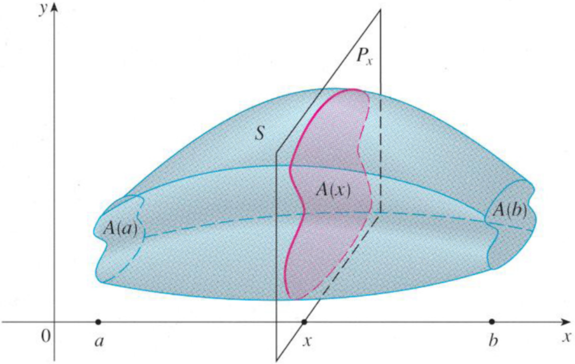
Section 4.2: Volumes using Cross-Sectional Area

Suppose we have a solid S that sitting along the x-axis from $x = a$ to b.



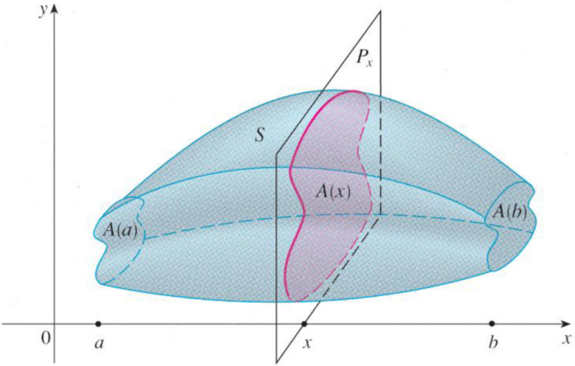
Section 4.2: Volumes using Cross-Sectional Area

We slice S using a plane P_x through x and perpendicular to the x-axis.



Section 4.2: Volumes using Cross-Sectional Area

Let $A(x)$ be the *cross-sectional area*.



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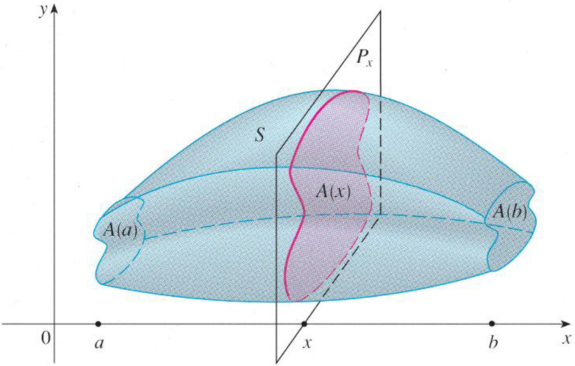
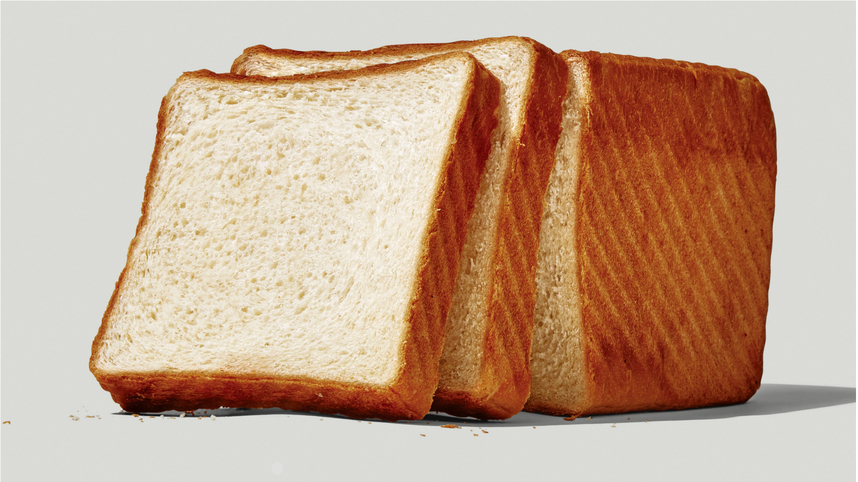


The volume of S is then given by

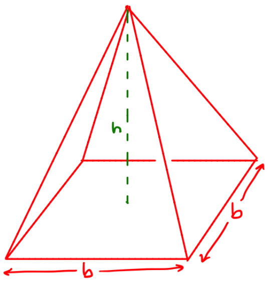
$$\mathrm{Vol}(S) = \int_a^b A(x)\,dx.$$

Volumes using Cross-Sectional Area



Example: Pyramids

Find the volume of a square pyramid of height h and base length b.



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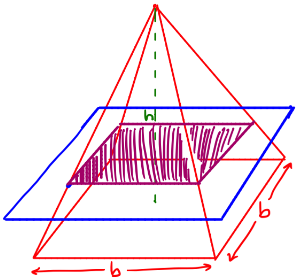

Slice the pyramid with a plane P_y parallel to the base of the pyramid and is y units above the base.

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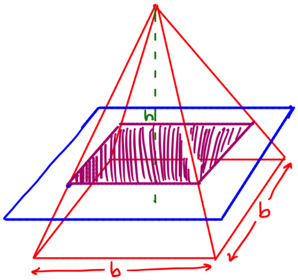
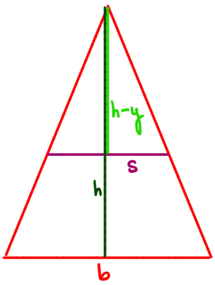


Slice the pyramid with a plane P_y parallel to the base of the pyramid and is y units above the base.

We get a square. What is its side length s?

Pyramids

Here's a side view of the pyramid:



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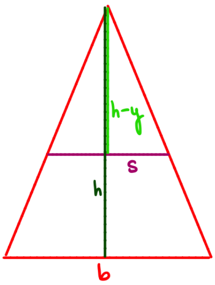

Because we see similar triangles, we know that

$$\frac{s}{b} = \frac{h - y}{h}.$$

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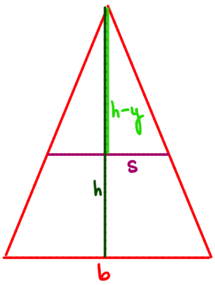

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Therefore,

$$s = \frac{b}{h}(h-y)$$

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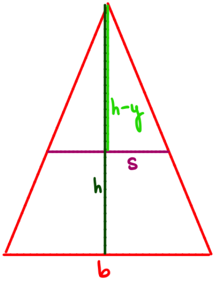

Because we see similar triangles, we know that

$$\frac{s}{b} = \frac{h-y}{h}.$$

Therefore,

$$s = \frac{b}{h}(h-y)$$

and the area of this cross-section is $A(y) = \frac{b^2}{h^2}(h-y)^2$.

Pyramids

Therefore the area of this slice is $A(y) = \dfrac{b^2}{h^2}(h - y)^2$.

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The volume of the pyramid is

$$\int_0^h A(y)\, dy = \int_0^h \frac{b^2}{h^2}(h-y)^2\, dy$$

Pyramids

Therefore the area of this slice is $A(y) = \frac{b^2}{h^2}(h - y)^2$.

The volume of the pyramid is

$$\int_0^h A(y)\,dy = \int_0^h \frac{b^2}{h^2}(h-y)^2\,dy$$
$$= \frac{b^2}{h^2}\int_0^h (h-y)^2\,dy$$

Pyramids

Therefore the area of this slice is $A(y) = \frac{b^2}{h^2}(h - y)^2$.
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$$\begin{aligned}\int_0^h A(y)\,dy &= \int_0^h \frac{b^2}{h^2}(h-y)^2\,dy \\ &= \frac{b^2}{h^2}\int_0^h (h-y)^2\,dy \\ &= -\frac{b^2}{h^2}\cdot\frac{1}{3}(h-y)^3\Big|_0^h\end{aligned}$$

Pyramids

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&= \frac{1}{3}b^2 h
\end{aligned}$$

Example: Volume of a Sphere

Consider a sphere of radius r along the x-axis with center at the origin.

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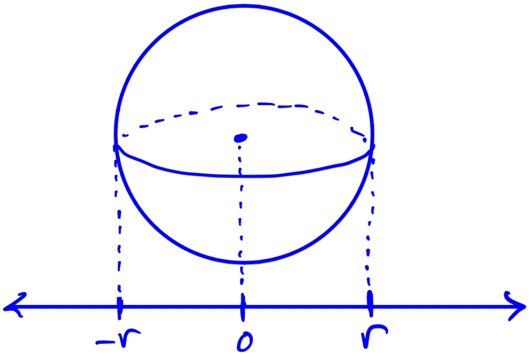
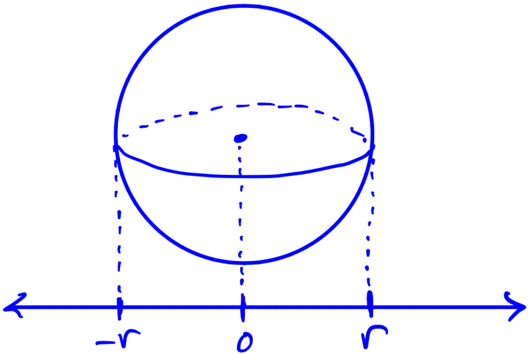


Example: Volume of a Sphere

Consider a sphere of radius r along the x-axis with center at the origin. Intersect the sphere with a plane through a point x.



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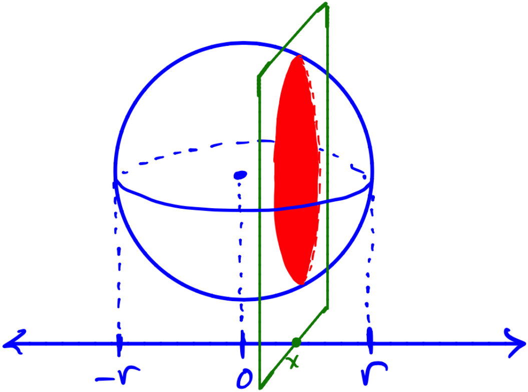

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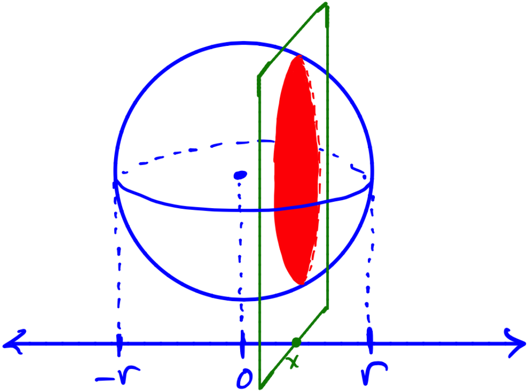

This gives a cross-section which is a circle.

Sphere

We want the area of this cross-section. Therefore, we want the radius.

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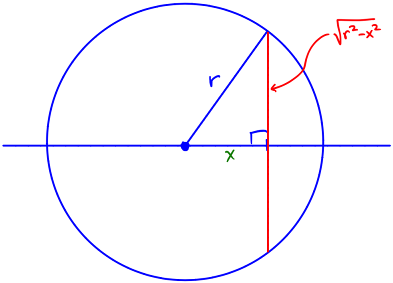

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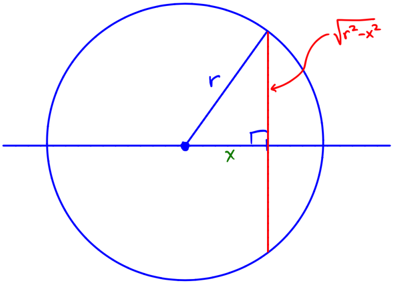

If R is the radius of the cross-section, then

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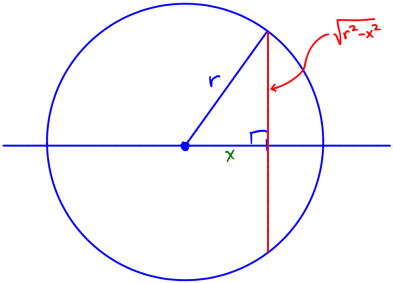

If R is the radius of the cross-section, then

$$x^2 + R^2 = r^2$$

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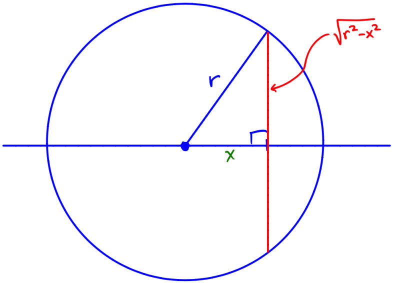

If R is the radius of the cross-section, then

$$x^2 + R^2 = r^2$$
$$R^2 = r^2 - x^2$$

Sphere

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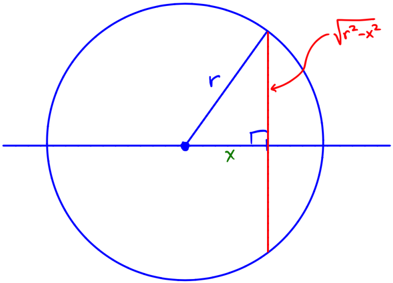

If R is the radius of the cross-section, then

$$\begin{aligned} x^2 + R^2 &= r^2 \\ R^2 &= r^2 - x^2 \\ R &= \sqrt{r^2 - x^2} \end{aligned}$$

Sphere

The area of this cross-section is $A(x) = \pi R^2$

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The volume of the sphere is

$$\int_{-r}^{r} A(x)\,dx = \pi \int_{-r}^{r} (r^2 - x^2)\,dx$$

Sphere

The area of this cross-section is $A(x) = \pi R^2 = \pi(r^2 - x^2)$.
The volume of the sphere is

$$\begin{aligned}\int_{-r}^{r} A(x)\,dx &= \pi \int_{-r}^{r} (r^2 - x^2)\,dx \\ &= 2\pi \int_{0}^{r} (r^2 - x^2)\,dx\end{aligned}$$

Sphere

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&= 2\pi \int_{0}^{r} (r^2 - x^2)\,dx \\
&= 2\pi \left(r^2 x - \frac{1}{3} x^3 \right)_0^r \\
&= 2\pi \left(r^3 - \frac{1}{3} r^3 \right)
\end{aligned}$$

Sphere

The area of this cross-section is $A(x) = \pi R^2 = \pi(r^2 - x^2)$.
The volume of the sphere is

$$\begin{aligned}
\int_{-r}^{r} A(x)\,dx &= \pi \int_{-r}^{r} (r^2 - x^2)\,dx \\
&= 2\pi \int_{0}^{r} (r^2 - x^2)\,dx \\
&= 2\pi \left(r^2 x - \frac{1}{3} x^3 \right)_{0}^{r} \\
&= 2\pi \left(r^3 - \frac{1}{3} r^3 \right) \\
&= \frac{4\pi}{3} r^3.
\end{aligned}$$

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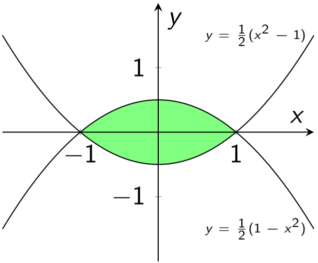

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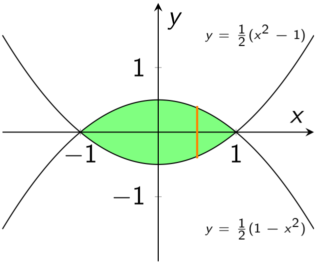

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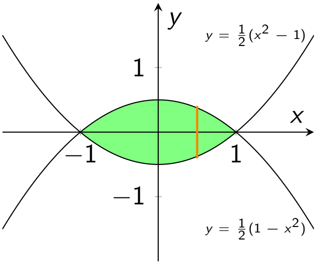

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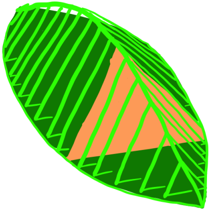

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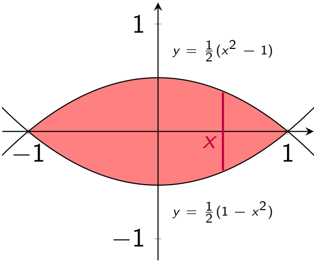

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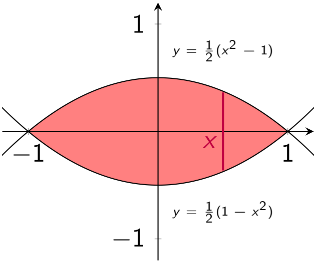

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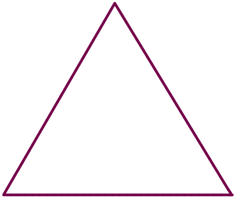

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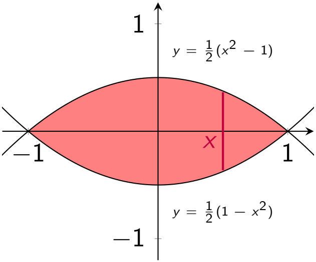

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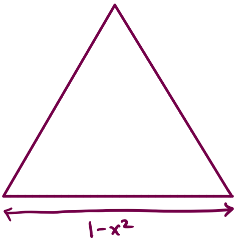

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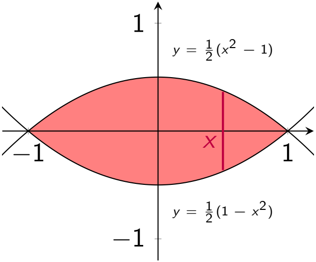

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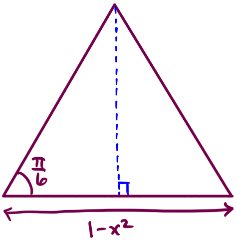

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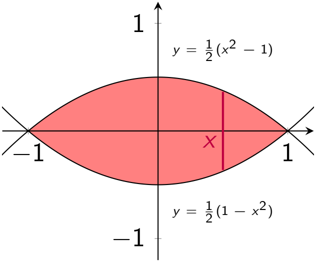

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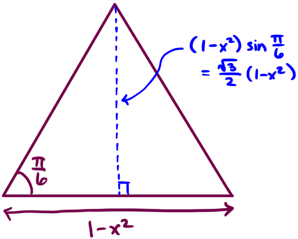

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