Homework 1

Due Wednesday, January 15, 2020

1. Calculate $\mathcal{U}(f,\mathcal{P})$ and $\mathcal{L}(f,\mathcal{P})$ for each of the following functions f and partitions $\mathcal{P}$ of $[a,b]$.
   (a) (3 points) $f(x) = \dfrac{x}{x+1}$ and $\mathcal{P} = \{0,1,2,3\}$ is a partition of $[0,3]$
   (b) (3 points) $f(x) = \sin(x)$ and $\mathcal{P} = \{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi\}$ is a partition of $[0,\pi]$
2. Let f be a function, continuous on $[-1,1]$ and take $\mathcal{P}$ to be a partition of $[-1,1]$. Show that each of the following three statements is false.
   (a) (2 points) $\mathcal{L}(f,\mathcal{P}) = 3$ and $\mathcal{U}(f,\mathcal{P}) = 2$
   (b) (2 points) $\mathcal{L}(f,\mathcal{P}) = 3$, $\mathcal{U}(f,\mathcal{P}) = 6$, and $\int_{-1}^{1} f = 2$
   (c) (2 points) $\mathcal{L}(f,\mathcal{P}) = 3$, $\mathcal{U}(f,\mathcal{P}) = 6$, and $\int_{-1}^{1} f = 10$
3. Recall that
   $$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$
   A *regular partition* of $[a,b]$ is a partition $\mathcal{P}$ of points $a = x_0 < x_1 < \cdots < x_n = b$ such that
   $$x_i - x_{i-1} = \frac{b-a}{n}$$
   for each $i = 1,\ldots,n$, i.e. the points of the partition are equally spaced.
   Let $f(x) = x^2$. For each n, let $\mathcal{P}_n$ be the regular partition of $[0,1]$ consisting of points
   $$\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$
   (a) (4 points) Calculate $\mathcal{U}(f,\mathcal{P}_n)$ and $\mathcal{L}(f,\mathcal{P}_n)$ for each n.
   (b) (4 points) Evaluate $\overline{\int_0^1} x^2\,dx$ and $\underline{\int_0^1} x^2\,dx$ by taking the limits as $n \to \infty$.