Midterm 1 Review

25 October 2018

1. Use Mathematical Induction to prove that

   (a) For any integer $n \geq 1$, if $0 \leq a < b$, then $a^n < b^n$.

   Base Case: If $n=1$, then $0 \leq a < b \implies a^1 < b^1$. Done.

   Inductive Step: Let $0 \leq a < b$. Assume that $a^n < b^n$. Then

   $$\begin{aligned} a^{n+1} &= a^n \cdot a && \text{(law of exponents)} \\ &< a^n \cdot b && (a<b) \\ &< b^n \cdot b && (a^n < b^n) \\ &= b^{n+1} && \text{(law of exponents)} \end{aligned}$$

   Therefore, the statement holds for all $n \geq 1$.

   (b) For any integer $n \geq 2$,

   $$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}.$$

   Base Case If $n=2$, then $\left(1-\frac{1}{2}\right)=\frac{1}{2}$. Done.

   Inductive Step If $\left(1-\frac{1}{2}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$,

   $$\begin{aligned} \text{then } \left(1-\frac{1}{2}\right)\cdots\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right) &= \frac{1}{n}\left(1-\frac{1}{n+1}\right) \\ &= \frac{1}{n}\left(\frac{n+1}{n+1}-\frac{1}{n+1}\right) \\ &= \frac{1}{n}\left(\frac{n}{n+1}\right) \\ &= \frac{1}{n+1}. \end{aligned}$$

   Therefore, by induction, the statement holds for all $n \geq 2$

   (c) For any integer $n \geq 2$,

   $$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

   Base Case For $n=2$, $\left(1-\frac{1}{2^2}\right)=\frac{3}{4}$. Done.

   Inductive Step If $\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$, then

   $$\begin{aligned} \left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) &= \frac{n+1}{2n}\left(1-\frac{1}{(n+1)^2}\right) \\ &= \frac{n+1}{2n}\left(\frac{(n+1)^2-1}{(n+1)^2}\right) \\ &= \frac{1}{2n}\cdot\frac{n^2+2n+1-1}{n+1} \\ &= \frac{n^2+2n}{2n(n+1)} \\ &= \frac{n+2}{2(n+1)} \end{aligned}$$

   Therefore, the equation holds for all $n \geq 2$.

2. Evaluate the following limits or indicate if they do not exist.

(a) $\lim_{x\to 1} \frac{x}{x+1}$

$\frac{x}{x+1}$ is a rational function. Therefore, it is continuous everywhere it is defined. Hence,

$\lim_{x\to 1} \frac{x}{x+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$.

(b) $\lim_{x\to 0} \frac{x(1+x)}{2x^2} = \lim_{x\to 0} \frac{1+x}{2x}$. Plug in $x=0$, get $\frac{1}{0}$. Therefore, limit does not exist.

(c) $\lim_{x\to 1} \frac{x^4-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^3+x^2+x+1)}{x-1}$

$= \lim_{x\to 1} (x^3+x^2+x+1)$

$= \boxed{4}$

(d) $\lim_{x\to 9} \frac{x-3}{\sqrt{x}-3}$. Plug in $x=9$, get $\frac{6}{0}$. Therefore, this limit does not exist.

(e) $\lim_{x\to 0} x\left(1-\frac{1}{x}\right) = \lim_{x\to 0} (x-1)$

$= 0-1$

$= \boxed{-1}$

(f) $\lim_{x \to -4} \left(\frac{2x}{x+4} + \frac{8}{x+4} \right)$ $= \lim_{x\to -4} \left(\frac{2x+8}{x+4} \right)$

$= \lim_{x\to -4} \frac{2(x+4)}{x+4}$

$= \lim_{x\to -4} 2$

$= \boxed{2}$

3. Using the $\epsilon - \delta$ definition of limits, prove that

(a) $\lim_{x \to 1} (x^2 - 5x + 2) = -2$

Let $\varepsilon > 0$. Pick $\delta = \min\{1, \frac{\varepsilon}{4}\}$. If $0 < |x-1| < \delta$, then

$|(x^2-5x+2)-(-2)| = |x^2-5x+4|$
$= |x-1||x-4|$
$< \delta |x-4|$
$= \delta |(x-1)+(1-4)|$
$\leq \delta(|x-1|+3)$
$< \delta(\delta+3)$
$\leq 4\delta$
$\leq \varepsilon$.

Therefore, by definition of limits, $\lim_{x\to 1} (x^2-5x+2) = -2$.

(b) $\lim_{x \to -1} (x^2 + 2x) = -1$

Let $\varepsilon > 0$. Pick $\delta = \min\{1, \frac{\varepsilon}{1}\}$. If $0 < |x-(-1)| < \delta$, then

$|(x^2+2x)-(-1)| = |x^2+2x+1|$
$= |x+1| \cdot |x+1|$
$< \delta \cdot \delta$
$\leq \delta \cdot 1$
$\leq \varepsilon$

Therefore, by the definition of limits, $\lim_{x\to -1} (x^2+2x) = -1$.

Note: $\delta = \sqrt{\varepsilon}$ would also work for this problem.

4. Given that

$$\lim_{x \to c} f(x) = 2 \qquad \lim_{x \to c} g(x) = -1 \qquad \lim_{x \to c} h(x) = 0$$

evaluate the limits that exist. If the limit does not exist, state why.

(a) $\lim_{x \to c} [f(x) - g(x)]$

$= \left(\lim_{x\to c} f(x)\right) - \left(\lim_{x\to c} g(x)\right)$

$= 2 - (-1)$

$= \boxed{3}$

(b) $\lim_{x \to c} [f(x)]^2$

$= \left(\lim_{x\to c} f(x)\right)^2$

$= (2)^2$

$= \boxed{4}$

(c) $\lim_{x\to c} \frac{f(x)}{g(x)}$ $= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{-1} = \boxed{-2}$

provided that $\lim_{x\to c} g(x) \neq 0$.

(d) $\lim_{x\to c} \frac{h(x)}{f(x)}$ $= \frac{\lim_{x\to c} h(x)}{\lim_{x\to c} f(x)} = \frac{0}{2} = \boxed{0}$

provided that $\lim_{x\to c} h(x) \neq 0$.

(e) $\lim_{x\to c} \frac{f(x)}{h(x)}$ $= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} h(x)} = \frac{2}{0}$. DNE.

nonzero

(f) $\lim_{x\to c} \frac{1}{f(x)-g(x)}$ $= \frac{1}{\lim_{x\to c}(f(x)-g(x))}$

provided $\lim_{x\to c}(f(x)-g(x)) \neq 0$.

$= \frac{1}{(\lim_{x\to c} f(x)) - (\lim_{x\to c} g(x))}$

$= \frac{1}{2-(-1)} = \boxed{1/3}$

5. Determine whether or not the function is continuous at the indicated point. Explain why.

(a) $f(x) = x^3 - 5x + 1$ at $x = 2$

Polynomials are continuous everywhere.

$\boxed{\text{Yes}}$

(b) $f(x) = \begin{cases} x^2+4, & x<2 \\ 5, & x=2 \\ x^3, & x>2 \end{cases}$ at $x=2$

1) f is defined at 2.

2) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (x^2+4) = 8$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^3) = 2^3 = 8$

$\Longrightarrow$ $\lim\limits_{x\to 2} f(x)$ exists and equals 8.

3) $\lim\limits_{x\to 2} f(x) = 8 \neq f(2) = 5 \longrightarrow$ Not Continuous

(c) $f(x) = \begin{cases} \frac{x^2-1}{x+1}, & x \neq -1 \\ -2, & x=-1 \end{cases}$ at $x=-1$

1) f is defined at $x=-1$.

2) $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \frac{x^2-1}{x+1}$

$= \lim\limits_{x\to -1} \frac{(x+1)(x-1)}{x+1}$

$= \lim\limits_{x\to -1} (x-1)$

$= -2$

3) $\lim\limits_{x\to -1} f(x) = 2 = f(-1) \longrightarrow$ Therefore, f is continuous at $x=-1$.

(d) $f(x) = \begin{cases} -x^2, & x<0 \\ 0, & x=0 \\ 1/x^2, & x>0 \end{cases}$ at $x=0$

1) $f(0)=0$ is defined

2) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{1}{x^2}$ does not exist $\Longrightarrow$ f not continuous at $x=0$.

6. Evaluate the limits or indicate if they do not exist.

(a) $\lim\limits_{x\to 0} \frac{\sin(4x)}{\sin(2x)} = \lim\limits_{x\to 0} \frac{\sin(4x)}{\sin(2x)} \cdot \frac{x}{x}$

$= \lim\limits_{x\to 0} \frac{\sin(4x)}{x} \cdot \frac{x}{\sin(2x)}$

$= \lim\limits_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{4}{2}$

$= 1 \cdot 1 \cdot 2$

$= \boxed{2}$

(b) $\lim_{x\to 0} \frac{\sin(x^2)}{x} = \lim_{x\to 0} \frac{\sin(x^2)}{x} \cdot \frac{x}{x}$

$= \lim_{x\to 0} \frac{\sin(x^2)}{x^2} \cdot x$

$= 1 \cdot 0$

$= \boxed{0}$

(c) $\lim_{x\to 0} \frac{\sin(1+x)}{1-x} = \frac{\sin(1)}{1} = \sin(1)$

by plugging in.

(d) $\lim_{x\to 0} \frac{2x}{\tan(3x)} = \lim_{x\to 0} \frac{2x}{\sin(3x)} \cdot \cos(3x)$

$= \lim_{x\to 0} \frac{3x}{\sin(3x)} \cdot \frac{2}{3} \cdot \cos(3x)$

$= 1 \cdot \frac{2}{3} \cdot 1$

$= \boxed{\frac{2}{3}}$

(e) $\lim_{x\to 0} \frac{\cos(x)-1}{2x} = -\frac{1}{2} \lim_{x\to 0} \frac{1-\cos(x)}{x}$

$= -\frac{1}{2} \cdot 0$

$= \boxed{0}$

(f) $\lim_{x\to 0} \frac{1-\cos(4x)}{3x} = \lim_{x\to 0} \frac{1-\cos(4x)}{4x} \cdot \frac{4}{3}$

$= 0 \cdot 4/3$

$= \boxed{0}$

(g) $\lim_{x\to -1} \frac{\cos(x+1)-1}{2(x+1)} = \lim_{y\to 0} \frac{\cos(y)-1}{2y} = \boxed{0}$ by (e).

substitute $y = x+1$.
as $x \to -1$, then $y \to 0$.

(h) $\lim_{x\to 0} \frac{\cos(2x)}{\cos(x)} = \frac{\lim_{x\to 0} \cos(2x)}{\lim_{x\to 0} \cos(x)} = \frac{\cos(0)}{\cos(0)} = 1.$

7. Use the Squeeze/Pinching Theorem to calculate

$$\lim_{x\to 0} x \sin^2(1/x).$$

We have $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

so $0 \le \sin^2\left(\frac{1}{x}\right) \le 1.$

We multiply by x. This gives 2 cases.

1) If $x \geq 0$, then

$0 \le x\sin^2\left(\frac{1}{x}\right) \le x = |x|$ (since $x \geq 0$ implies $x = |x|$)

2) If $x \le 0$, then

$0 \geq x\sin^2\left(\frac{1}{x}\right) \geq x. = -|x|$ (since $x \le 0$ implies $x = -|x|$)

Thus, we have

$$-|x| \le x\sin^2\left(\frac{1}{x}\right) \le |x|$$

for all x.

Then $\lim_{x\to 0}(-|x|) = \lim_{x\to 0} |x| = 0$ so

by the Squeeze Theorem, $\lim_{x\to 0} x\sin^2\left(\frac{1}{x}\right) = 0.$

8. Use the Squeeze/Pinching Theorem to calculate

$$\lim_{x\to\pi} \left((x-\pi)\cos^2\left(\frac{1}{x-\pi}\right) \right).$$

We have $-1 \le \cos\left(\frac{1}{x-\pi}\right) \le 1$

So $0 \le \cos^2\left(\frac{1}{x-\pi}\right) \le 1.$

Multiplying by $(x-\pi)$, we have 2 cases:

1) If $x-\pi \geq 0$, then

$0 \le (x-\pi)\cos^2\left(\frac{1}{x-\pi}\right) \le (x-\pi) = |x-\pi|$

2) If $x-\pi \le 0$, then

$0 \geq (x-\pi)\cos^2\left(\frac{1}{x-\pi}\right) \geq (x-\pi) = -|x-\pi|$

Thus, $-|x-\pi| \le (x-\pi)\cos^2\left(\frac{1}{x-\pi}\right) \le |x-\pi|.$

Since $\lim_{x\to\pi}(-|x-\pi|) = \lim_{x\to\pi}|x-\pi| = 0$, by the Squeeze Theorem,

$\lim_{x\to\pi} (x-\pi)\cos^2\left(\frac{1}{x-\pi}\right) = 0.$

9. Use the Intermediate Value Theorem to show that there is a solution of the given equation in the indicated interval.

(a) $2x^3 - 4x^2 + 5x - 4 = 0; [1, 2]$

Let $f(x) = 2x^3 - 4x^2 + 5x - 4$. Then f is continuous everywhere, so f is continuous on $[1,2]$. We have

$f(1) = 2 - 4 + 5 - 4 = -1$

$f(2) = 2(8) - 4(4) + 5(2) - 4 = 16 - 16 + 10 - 4 = 6$

Since 0 is between -1 and 6, by the IVT, there is some c such that $f(c) = 0$ and c lies in $(1,2)$.

(b) $\sin(x) + 2\cos(x) - x^2 = 0; [0, \pi/2]$

Let $f(x) = \sin(x) + 2\cos(x) - x^2$. Then f continuous everywhere, in particular on $[0, \pi/2]$. We have

$$f(0) = 0 + 2 - 0^2 = 2$$
$$f(\pi/2) = 1 + 0 - \left(\frac{\pi}{2}\right)^2 = 1 - \frac{\pi^2}{4} < 0.$$

Since 0 lies between 2 and $1 - \frac{\pi^2}{4}$, there is some c such that $f(c) = 0$. (by the IVT).

(c) $x^3 = \sqrt{x+2}; [1, 2]$

Let $f(x) = x^3 - \sqrt{x+2}$. Then $f(x)$ is continuous on $(-2, \infty)$ so it is continuous on $[1,2]$. Then

$$f(1) = 1^3 - \sqrt{1+2} = 1 - \sqrt{3} < 0$$
$$f(2) = 2^3 - \sqrt{2+2} = 8 - 4 = 4 > 0$$

By the IVT, there is some c in $[1,2]$ such that

$$f(c) = 0$$
$$c^3 - \sqrt{c+2} = 0$$
$$c^3 = \sqrt{c+2}$$

(d) $2\tan(x) - x = 1; [0, \pi/4]$

Let $f(x) = 2\tan(x) - x$. This is continuous whenever at all points $\cos(x) \neq 0$, since $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

In particular, it is continuous on $[0, \pi/4]$. Then,

$$f(0) = 2\tan(0) - 0 = 0$$
$$f\left(\frac{\pi}{4}\right) = 2\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}$$
$$= 2 \cdot 1 - \frac{\pi}{4}$$
$$= 2 - \pi/4 > 1 \quad \text{since } \frac{\pi}{4} < 1$$
$$-\frac{\pi}{4} > -1$$
$$2 - \pi/4 > 2 - 1 = 1.$$

Therefore, there is some c in $[0, \pi/4]$ such that $f(c) = 1$.

$$2\tan(c) - c = 1.$$