Chapter 1: PreCalc Review

Oct 1

- inequalities
- functions
  - elementary functions
  - operations on functions
- mathematical proofs
  - induction

Inequalities (§1.3)

Notation
$>$ "(strictly) greater than" $<$ "(strictly) less than"
$\geq$ "greater than or equal to" $\leq$ "less than or equal to"

- x is positive $(x>0)$
  negative $(x<0)$
  nonnegative $(x\geq 0)$
  nonpositive $(x\leq 0)$
- interval notation
  (a,b) (a,∞)
  $[a,b)$ $[a,\infty)$
  $(a,b]$ $(-\infty,b)$
  $[a,b]$ $(-\infty,b]$
- set notation $\{x : \text{some property of } x \text{ holds}\}$
  $\in$ $\subseteq$ $\subsetneq$ $\cup$

Solving Inequalities

Keep Inequality
- adding any number
- multiply by POSITIVE number

Reverse Inequality
- multiply by NEGATIVE number
- taking reciprocals

Example Solve a linear inequality

Absolute Value: Given a number x, we let

$$|x| = \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x\leq 0 \end{cases}$$

(Alternatively, $|x| = \sqrt{x^2}$ • the $\sqrt{\ }$ function is always positive.)

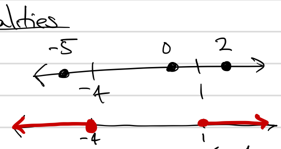
Example $|5| = 5$ $|-202| = 202$
$|-3| = 3$ $|0| = 0$

Quadratic Formula: Solve $ax^2+bx+c=0$ for x where a,b,c are numbers.

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

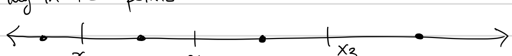
General Method for Solving Inequalities

Example $x^2+3x-4\geq 0$
$x^2+3x-4 = 0$
$(x+4)(x-1) = 0$
$x = 1, -4$



-5 ✓, 0 ✗, 2 ✓ $\Rightarrow$ $x\leq -4$ or $x\geq 1$

1. Replace inequality by $=$.
2. Solve new inequality for solutions $x_1, x_2, x_3, \ldots, x_n$.
3. Plug in test points



4. Plot on number line. ($\circ$ = endpoint not included, $\bullet$ = endpoint included)
5. Write down solution.

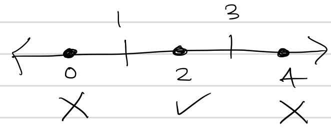
Example $(x+3)(x-3)(x+4)<0$.

The absolute value $|a|$ can be thought of as the "distance from 0 to a".

The value $|a-b|$ can be thought of as the "distance from a to b."

Example $|x-2|<1$

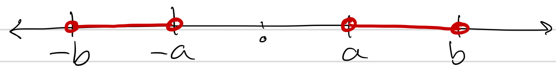
1) $|x-2| = 1$ ($|a| = 1$ iff $a = \pm 1$)
$x-2 = \pm 1$
1: $x = 3$ -1: $x = 1$



Solution: $1<x<3$

In general, we can replace
- **"$|x|<c$" by "$-c<x<c$".**
- **"$|x|>c$" by "$x<-c$ or $x>c$".**

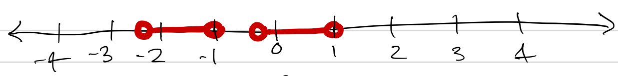
Example $a<|x|<b$
$\Updownarrow$
$a<|x|$ and $|x|<b$



"$a<|x|<b$" is equivalent to "$-b<x<-a$ or $a<x<b$."

Example Solve $1<|3x+2|<5$

$-5<3x+2<-1$ OR $1<3x+2<5$
$-7<3x<-3$ OR $-1<3x<3$
$-\frac{7}{3}<x<-1$ OR $-\frac{1}{3}<x<1$



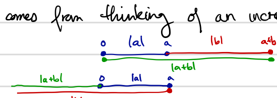
$(-\frac{7}{3},-1)\cup(-\frac{1}{3},1)$

Triangle Inequality Let a,b be (real) numbers. Then

$$|a+b|\leq |a|+|b|.$$

Equality holds if and only if a and b have the same sign.

"Triangle" comes from thinking of an incredibly thin "triangle".



In "$|a+b|$" p... es and negatives can cancel. In "$|a|+|b|$" this cannot happen.

Functions (§1.5)

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What is a function?

A function f comes with the following information.
1) domain — set of possible inputs $\mathrm{Dom}(f)$
2) an assignment of a single output to each input (an input cannot have >1 output). $f(x)$
3) range — set of possible outputs $\mathrm{Range}(f)$

Since the range is the set of values obtained by applying f to elements of $\mathrm{Dom}(f)$, only need $\mathrm{Dom}(f)$ and the assignment $f(x)$ to specify a function.

Examples 1) $f(x) = x^2$ with domain the set $(-\infty,\infty)$.
Its range is then the set $[0,\infty)$
2) $g(x) = x^2$ with domain the set $[0,\infty)$
Its range is then $[0,\infty)$.

Note f and g are not the same function since their domains are not equal.

Example 1) The function f defined on $(0,1)$ by $f(x) = \frac{1}{x(1-x)}$.
2) The function g defined on $(-\infty,\infty)$ by $g(x) = |x|$.

Piecewise Functions

We can also split a domain into 2 or more "pieces" and define the assignment differently on each piece.

E.g. f defined on $(-\infty,\infty)$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0,\infty) \\ -x & \text{if } x \in (-\infty, 0] \end{cases}$$

We just need to make sure that the assignments agree on the overlap of two or more pieces.

E.g. g defined on $(-\infty,\infty)$ by $g(x) = \begin{cases} x^2 & \text{if } x \in [0,\infty) \\ -x & \text{if } x \in (-\infty, 1] \end{cases}$.

This is NOT a function since $g(1)$ has two different values 1 and -1.

We say that g is not well-defined

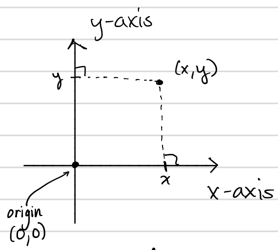
Example Is the relation f defined on $[0,3]$ by

$$f(x) = \begin{cases} x^2-3 & x \in [0,1] \\ x-3 & x \in [1,2] \\ |x|-3 & x \in [1,3] \end{cases}$$

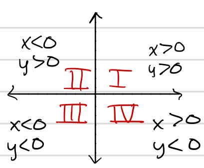
a function?

Graph of a Function

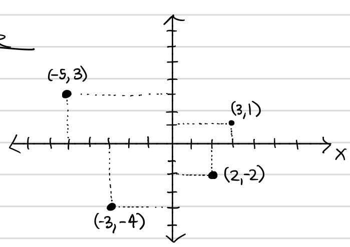
Coordinate Plane:



Four quadrants:



Example



The graph of a function f is the subset of the coordinate plane given by pairs $(x, f(x))$ for $x \in \mathrm{Dom}(f)$.

The Elementary Functions (§1.6)

1) Polynomials Obtained from $\cdot$ and $+$

Examples $f(x) = 2$
$g(x) = x+3$
$h(x) = 2x^3 - x + 5$

These have assignments given by
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$
where $a_0, \dots, a_n$ are (real) numbers (called the coefficients) and $a_n \neq 0$ (called the leading coefficient)

Examples constants, linear, quadratics, etc.

2) Rational Functions obtained from $\cdot$, $+$, and $\div$.

A rational function has an assignment of the form
$$f(x) = \frac{p(x)}{q(x)}$$
where p and q are polynomials.

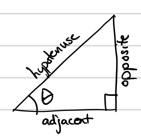
Such a rational function can be defined wherever $q(x) \neq 0$.

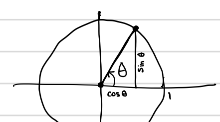
Examples: $f(x) = \frac{1}{1+x^2}$

$$g(x) = \frac{x^2 - x + 1}{x+1}$$

Example: What is the domain of f and g?

3) Trig Functions





θ is measured in radians

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Other trig functions: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

Particular Values
* Learn these
Identities
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Compositions of Functions (§1.7)

Let f and g be functions. New functions $f+g$, $f-g$, fg, $\frac{f}{g}$, cf.

Define a new function $f \circ g$, with domain the set of all x such that $g(x) \in \mathrm{Dom}(f)$, by
$$(f \circ g)(x) = f(g(x))$$

Example $f(x) = |x+3|$
$g(x) = \frac{1}{x-3}$

$$\rightsquigarrow (f \circ g)(x) = f(g(x)) = |g(x)+3| = \left|\frac{1}{x-3} + 3\right|$$

Mathematical Induction (§1.8) Oct 5

A theorem is a true statement. E.g. If $\underbrace{x+3=1}_{\text{hypothesis}}$, then $\underbrace{x=-2}_{\text{conclusion}}$.

A proof is a "rigorous" argument for why the theorem is true.

E.g. Proof that "if $3x > x+1$, then $x > \frac{1}{2}$"

$3x > x+1$
$\Rightarrow 2x > 1$ (subtract x from both sides)
$\Rightarrow x > \frac{1}{2}$ (divide by 2 on both sides)

($\Rightarrow$ means "implies")

Suppose A and B are statements
- $A \Rightarrow B$ means "If A, then B." or "A implies B"
- $A \Leftrightarrow B$ means "A if and only if B."

$B \Rightarrow A$ and $A \Rightarrow B$

Converse of $A \Rightarrow B$ is $B \Rightarrow A$.
Contrapositive of $A \Rightarrow B$ is B is false $\Rightarrow$ A is false.

Examples 1) All dogs are mammals.

$\Updownarrow$

If x is a dog, then x is a mammal.

Converse: If x is a mammal, then x is a dog.
Contrapositive: If x is not a mammal, then x is not a dog.

2) If $x=1$, then $x^2=1$.

Converse: If $x^2=1$, then $x=1$.
Contrapositive: If $x^2 \neq 1$, then $x \neq 1$.

Proof by Contradiction

An implication is equivalent to its contrapositive:
- "$A \Rightarrow B$" is equivalent to "B is false $\Rightarrow$ A is false"

Proof by Contradiction: Suppose we want to prove "$A \Rightarrow B$."

PbC says 1st assume "A is true and B is false."
Next, crank the machine until you get a contradiction like "$1 \neq 1$" or "$0=1$".

Example Prove that $\sqrt{2}$ is irrational. (not rational)

That is, $x=\sqrt{2} \Rightarrow x$ is irrational.

1) Assume $x=\sqrt{2}$ and x is rational, so has the form $x=\frac{p}{q}$ for integers p,q.

We may assume that $\frac{p}{q}$ is reduced.

2) $x^2=2$
$\Rightarrow \frac{p^2}{q^2}=2$
$\Rightarrow p^2=2q^2$

so 2 divides p^2 so 2 must divide p. Write $p=2p'$

Then, $\frac{(2p')^2}{q^2}=2 \Rightarrow 4(p')^2=2q^2 \Rightarrow 2(p')^2=q^2$

So, 2 also divides q^2 and so 2 divides q.

This contradicts the fact that we may write $\frac{p}{q}$ so that p and q have no factors in common.

Mathematical Induction is a form of proof.

Let $P(n)$ be a statement about an integer n

Ex. $P(n)=$ "n^2+1 is odd"
$P(n)=$ "n^2-n is divisible by 3."

Suppose we want to prove that $P(n)$ is true for all positive integers.

Need:
- $P(1)$ is true "base case"
- $P(n)$ is true $\Rightarrow P(n+1)$ is true "inductive step"

Then $P(n)$ is true for all positive integers

$P(1)$ true $\Rightarrow P(2)$ true $\Rightarrow P(3)$ true $\Rightarrow P(4)$ true $\Rightarrow \dots$
$\dots \Rightarrow P(10000005)$ true $\Rightarrow \dots$

Similar to dominoes.

Example Show that for all integers $n \geq 1$, $n \leq n^2$.

1) Base case: $1 \leq 1$ ✓

2) Inductive step: $n \leq n^2$ ← Start
$\Rightarrow 0 \leq 2n \leq n^2+n$
$\Rightarrow 0 \leq n^2+n$
$\Rightarrow n+1 \leq n^2+2n+1$
$\Rightarrow n+1 \leq (n+1)^2$ ← Finish

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Example Prove that $2n \leq 2^n$ for all positive n

Base case: For $n=1$, $2 \leq 2$ is true ✓

Inductive Step: If $2n \leq 2^n$ is true, we want to show $2(n+1) \leq 2^{n+1}$.

Because n is positive, $2 \leq 2^n$.

$$\begin{array}{rl} & 2n \leq 2^n \\ + & 2 \leq 2^n \\ \hline & 2n+2 \leq 2^n+2^n \end{array}$$
$\Rightarrow 2(n+1) \leq 2 \cdot 2^n$
$\Rightarrow 2(n+1) \leq 2^{n+1}$ ✓

Adding $2 \leq 2^n$ and $2n \leq 2^n$, get $2(n+1) \leq 2^{n+1}$. ∎

Example Prove that $\overbrace{1+3+5+7+\cdots+(2n-1)}^{\text{sum of first } n \text{ odd numbers}} = n^2$ for all $n \geq 1$

1) Base case: $n=1$ $1=1^2$ ✓

Inductive Step:

If $1+3+\cdots+(2n-1)=n^2$, we want to show that
$\underbrace{1+3+\cdots+(2n-1)}_{=n^2}+(2(n+1)-1)=(n+1)^2$

$n^2+(2(n+1)-1) = n^2+2n+1 \overset{\checkmark}{=} (n+1)^2$

Recall: Graphs of Functions

Chapter 2: Limits and Continuity

<u>Question</u>: Given a function $f(x)$, what does $f(x)$ approach as x approaches c (but not equal to c)?

If $f(x)$ approaches the value L, write
$$\lim_{x\to c} f(x) = L$$
or
$$f(x) \to L \text{ as } x \to c.$$

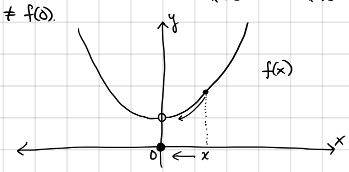
In general, $\lim_{x\to c} f(x) \neq f(c)$. The value $\lim_{x\to c} f(x)$ is completely independent of the value of $f(c)$.

<u>Example</u> $f(x) = 3x+2$.
As $x\to 0$, $3x \to 0$ and $3x+2 \to 2$.
So $\lim_{x\to 0}(3x+2) = 2$.

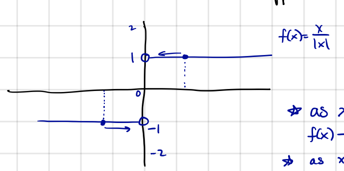
<u>Example</u> $f(x) = \frac{x}{x}$.
For any value $x\neq 0$, $f(x) = 1$. So as $x\to 0$, $f(x)$ is constantly 1, so $\lim_{x\to 0}\frac{x}{x} = 1$.

<u>Example</u> $f(x) = \begin{cases} x^2+1, & \text{if } x\neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Here, $f(0) = 0$.
As $x\to 0$, $f(x) = x^2+1$. Thus, $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2+1) = 1$.
So, $\lim_{x\to 0} f(x) \neq f(0)$.



<u>Example</u> $f(x) = \frac{x}{|x|}$. As $x\to 0$, $f(x)$ does not approach a definitive value.



* as $x\to 0$ from right, $f(x) \to 1$
* as $x\to 0$ from left, $f(x) \to -1$

<u>Left and Right-handed Limits</u>

$\lim_{x\to c^+} f(x) = L$: as $x\to c$ from the right, $f(x)\to L$

$\lim_{x\to c^-} f(x) = L$: as $x\to c$ from the left, $f(x)\to L$

<u>Example</u> $\lim_{x\to 0^+}\frac{x}{|x|} = 1$ and $\lim_{x\to 0^-}\frac{x}{|x|} = -1$

<u>Definition of Limit</u> (§2.2) Oct 10

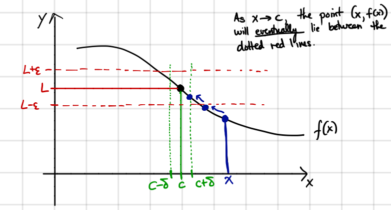
<u>Definition</u> We write $\lim_{x\to c} f(x) = L$ if

for all $\varepsilon > 0$, there is a $\delta > 0$ such that
$$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

- Instead of using our intuition, we now have a rigorous definition that we can use to evaluate limits.
- The δ and ε can be thought of as very small positive numbers.
- δ does not depend on x, only on ε.

$\lim_{x\to c} f(x)$ exists : there is an L such that $\lim_{x\to c} f(x) = L$.
$\lim_{x\to c} f(x)$ does not exist : there is no L such that $\lim_{x\to c} f(x) = L$

- We can always assume $\delta \leq$ constant. (if definition works for some $\delta > 0$, then it will also work for all $0 < \delta' \leq \delta$.)



If $x\in(c-\delta, c+\delta)$, then $f(x)\in(L-\varepsilon, L+\varepsilon)$.

- No matter how close to L we want (within ε), we want every x that is sufficiently close (within δ) to c to have the property that $f(x)$ is that close to L.
- In other words, if x is close to c, then $f(x)$ doesn't get too far from L.

<u>Proving Limits using ε-δ Definition</u>

Your proof should always start with "Let $\varepsilon > 0$. Pick $\delta = \ldots$"

Choosing the correct δ (which will be some function of ε) is the crux of the proof.

<u>Example</u> Prove that $\lim_{x\to 0}(3x+2) = 2$.

As per the definition, we want to (for any $\varepsilon > 0$) pick a δ such that
$$\underbrace{0 < |x-0|}_{=|x|} < \delta \implies \underbrace{|(3x+2)-2|}_{=3|x|} < \varepsilon$$

<u>Proof</u> Let $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{3}$. If $0 < |x-0| < \delta$, then $|x| < \delta$ and
$$|(3x+2)-2| = |3x| = 3|x| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon,$$
so $|(3x+2)-2| < \varepsilon$. Thus,
$$0 < |x-0| < \delta \implies |(3x+2)-2| < \varepsilon$$ ■

<u>Example</u> Prove that $\lim_{x\to 0}\frac{x}{x} = 1$.

<u>Proof</u> Pick any $\varepsilon > 0$. Let $\delta = \varepsilon$. Then
$$0 < |x-0| < \delta \implies x \neq 0$$
$$\implies \frac{x}{x} = 1$$
$$\implies \left|\frac{x}{x} - 1\right| = 0 < \varepsilon$$
$$0 < |x-0| < \delta \implies \left|\frac{x}{x} - 1\right| < \varepsilon.$$ ■

<u>Example</u> Prove that $\lim_{x\to 1}(-2x+3) = 1$.

<u>Proof</u> Let $\varepsilon > 0$. Pick $\delta = \frac{\varepsilon}{2}$. Then
$$0 < |x-1| < \delta \implies 2|x-1| < 2\delta$$
$$\implies 2|x-1| < \varepsilon$$
$$\implies |-2x+2| < \varepsilon$$
$$\implies |(-2x+3)-1| < \varepsilon$$ ■

<u>Example</u> Prove that $\lim_{x\to 0}(x^2-1) = -1$

$$0 < |x| < \delta \implies \underbrace{|(x^2-1)-(-1)|}_{=|x^2| = |x|^2} < \varepsilon$$

<u>Proof</u> Let $\varepsilon > 0$. Pick $\delta = \sqrt{\varepsilon}$. Then
$$0 < |x-0| < \delta \implies |x| < \delta$$
$$\implies |x|^2 < \delta^2$$
$$\implies |x^2| < \varepsilon$$
$$\implies |(x^2-1)-(-1)| < \varepsilon$$ ■

For this example, $\delta = \min\{1, \varepsilon\}$ will also work.

Example (Quadratic) Prove that $\lim_{x\to 1}(x^2-5x) = -4$.

Proof Let $\varepsilon > 0$. Let $\delta = \min\{1, \frac{\varepsilon}{4}\}$. Then $\delta \le 1$.

If $0 < |x-1| < \delta$, then

$$
\begin{aligned}
|(x^2-5x)-(-4)| &= |x^2-5x+4| && \text{(simplify)}\\
&= |(x-4)(x-1)| && \text{(factor)}\\
&= |x-4|\cdot|x-1| && \text{(use that } |ab| = |a|\cdot|b|\text{)}\\
&< |x-4|\cdot\delta && (|x-1|<\delta \text{ by assumption})\\
&= |x-1+1-4|\cdot\delta && \text{(subtract and add 1 inside the absolute value)}\\
&\le (|x-1|+|1-4|)\cdot\delta && \text{(triangle inequality)}\\
&< (\delta+3)\delta && \text{(simplify and use } |x-1|<\delta\text{)}\\
&\le 4\delta && (\text{use } \delta\le 1 \text{ so } \delta+3\le 4)\\
&\le \varepsilon && (\delta = \min\{1, \tfrac{\varepsilon}{4}\}, \text{ so } \delta \le \tfrac{\varepsilon}{4})
\end{aligned}
$$

Thus, $|(x^2-5x)-(-4)| < \varepsilon$ and

this proves $\lim_{x\to 1}(x^2-5x) = -4$.

Calculating Limits

Theorem Let k be a constant.

1) $\lim_{x\to c} k = k$
2) $\lim_{x\to c} x = c$.

Proof 1) Let $\varepsilon>0$. Pick $\delta = 1$. If $0<|x-c|<1$, then $|k-k| = 0 < \varepsilon$.

2) Let $\varepsilon>0$. Pick $\delta=\varepsilon$. If $0<|x-c|<\delta$, then $|x-c|<\varepsilon$.

Theorem (cont.) Let f, g be functions defined on an open interval containing c. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

3) $\lim_{x\to c}[k\cdot f(x)] = k\cdot[\lim_{x\to c} f(x)]$
4) $\lim_{x\to c}[f(x)+g(x)] = [\lim_{x\to c} f(x)] + [\lim_{x\to c} g(x)]$
5) $\lim_{x\to c}[f(x)\cdot g(x)] = [\lim_{x\to c} f(x)]\cdot[\lim_{x\to c} g(x)]$

Additionally, if $\lim_{x\to c} g(x) \ne 0$, then

6) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

Example 1)
$$
\begin{aligned}
\lim_{x\to 2}(x^2-5x) &\overset{\text{by } 4}{=} (\lim_{x\to 2} x^2) + (\lim_{x\to 2}(-5x))\\
&\overset{\text{by } 3}{=} (\lim_{x\to 2} x^2) + (-5)(\lim_{x\to 2} x)\\
&\overset{\text{by } 5}{=} (\lim_{x\to 2} x)(\lim_{x\to 2} x) + (-5)(\lim_{x\to 2} x)\\
&\overset{\text{by } 2}{=} (2)(2) + (-5)(2)\\
&\overset{\text{arithmetic}}{=} -6
\end{aligned}
$$

2) If $f(x)$ is a polynomial, then $\lim_{x\to c} f(x) = f(c)$.

3) If $f(x) = \frac{p(x)}{q(x)}$ is a rational function such that $q(c)\ne 0$, then $\lim_{x\to c} f(x) = f(c)$

Calculating Limits of Rational Functions

Let $f(x) = \frac{p(x)}{q(x)}$.

Goal: calculate $\lim_{x\to c} f(x)$.

Cases:

1) $p(c) = 0$ and $q(c)\ne 0$.

Then $\boxed{\lim_{x\to c}\frac{p(x)}{q(x)} = 0.}$

2) $p(c)\ne 0$ and $q(c)\ne 0$.

Then $\boxed{\lim_{x\to c}\frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.}$

3) $p(c)\ne 0$ and $q(c) = 0$

Then $\boxed{\lim_{x\to c}\frac{p(x)}{q(x)} \text{ does not exist.}}$

4) $p(c) = 0$ and $q(c) = 0$

May or may not exist.

Example $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

This example falls under case #4.

Notice that if $x\ne 1$, then $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$.

Since this limit doesn't consider when $x=1$, we may evaluate

$$
\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = \boxed{2}
$$

General Procedure: Evaluate $\lim_{x\to c}\frac{p(x)}{q(x)}$ with $p(c) = q(c) = 0$.

Step 1 : Factor $p(x)$ and $q(x)$.

Step 2 : Cancel all like factors.

Step 3 : Final result should lie in cases 1-3 above.

Example: $\lim_{x\to 2}\frac{x^2-3x+2}{x^2-x-2}$

Plug in $x=2$ to numerator and denominator, both become 0.

Factor and cancel:

$$
\lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+1)} = \lim_{x\to 2}\frac{x-1}{x+1} = \boxed{\frac{1}{3}}
$$

Continuity

Intuitively, a function is continuous if its graph can be drawn without picking up your pen.

More formally,

Definition A function f is continuous at c if the following 3 conditions hold:

1) f is defined in an open interval $(c-\delta, c+\delta)$ for some small $\delta>0$.
2) $\lim_{x\to c} f(x)$ exists (therefore, $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ must exist and be equal).
3) $\lim_{x\to c} f(x) = f(c)$.

If any of these fails to hold, we say that f is discontinuous at c.

Thus, for continuous functions, calculating limits is as easy as plugging in.

Definition A function is continuous on a set I if f is continuous at each point $c\in I$.

Examples 1) Polynomials are continuous on $(-\infty,\infty)$, where we say "continuous everywhere".

Thus, to evaluate $\lim_{x\to c} p(x)$ for $p(x)$ a polynomial, we have $\lim_{x\to c} p(x) = p(c)$.

2) Rational functions are continuous where they are defined.

3) $\sin(x)$ and $\cos(x)$ are continuous everywhere.

4) $f(x) = |x|$ is continuous everywhere.

5) $f(x) = \sqrt{x}$ is continuous on $(0,\infty)$ but not at 0 because f is not defined just to the left of 0.

We wish to include $f(x)=\sqrt{x}$ (at 0) in our definition of continuity. Afterall, its graph can be drawn without picking your pen up. For this, we define

Definition A function f is

A) right-continuous at c if

1) f is defined on $[c, c+\delta)$ for some $\delta>0$.
2) $\lim_{x\to c^+} f(x)$ exists
3) $\lim_{x\to c^+} f(x) = f(c)$.

B) left-continuous at c if

1) f is defined on $(c-\delta, c]$ for some $\delta>0$.
2) $\lim_{x\to c^-} f(x)$ exists
3) $\lim_{x\to c^-} f(x) = f(c)$.

Using this definition, we can say that "$f(x)=\sqrt{x}$ is right-continuous at 0."

A function is continuous on $[a,b]$, we mean that

1) f is continuous on (a,b)
2) f is right-continuous at a.
3) f is left-continuous at b.

Similar definitions hold for continuity on $(-\infty,b]$ and $[a,\infty)$.

Nonexamples 1) $f(x) = \begin{cases} x+1, & x\geq 0 \\ x^2, & x<0 \end{cases}$ is not continuous at $x=0$, because $\lim_{x\to 0} f(x)$ does not exist.

Recall: the limit $\lim_{x\to c} f(x)$ if and only if $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$.

Such a discontinuity is called a jump discontinuity

2) Define $f(x) = \frac{x^2-1}{x-1}$. This is not continuous at $x=1$ since $f(1)$ is not defined.

However, we can redefine $f(x)$ at $x=1$ so that it becomes continuous. (setting $f(1)=2$).

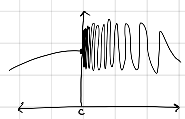
Such a discontinuity is called removeable.

In general, there are 3 types of discontinuities:

1) oscillations and asymptotes: either $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ does not exist.
2) jump: $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exists but are not equal.

"nonremovable"

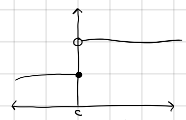
3) removable: $\lim_{x\to c} f(x)$ exists but is not equal to $f(c)$.



oscillation discontinuity

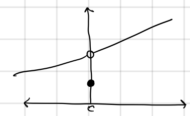
$\left(\lim_{x\to c^+} f(x) \text{ DNE.}\right)$

E.g. $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$



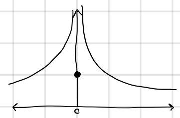
jump discontinuity

E.g. $f(x) = \begin{cases} 1, & x\leq 0 \\ -1, & x>0 \end{cases}$



removable discontinuity

E.g. $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x\neq 1 \\ 0, & x=1 \end{cases}$



asymptotic discontinuity

E.g. $f(x) = \begin{cases} \frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Example Where is $f(x) = \begin{cases} \frac{x^2+1}{x-2}, & x>0 \\ 3, & x\leq 0 \end{cases}$ continuous?

We break this problem into peices:

On $(0,\infty)$, $f(x) = \frac{x^2+1}{x-2}$. This is continuous everywhere except at $x=2$.

On $(-\infty, 0]$, $f(x)=3$. This is continuous everywhere except at $x=0$ (because it is the endpoint of a closed interval)

Therefore $f(x)$ is continuous at least on $(-\infty,0)\cup(0,2)\cup(2,\infty)$.

The question remains: is $f(x)$ continuous at $x=0$?

No, because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+1}{x-2} = -\frac{1}{2}$

and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3 = 3$

which are not equal.

Therefore, $f(x)$ is continuous exactly on $(-\infty,0)\cup(0,2)\cup(2,\infty)$.

Squeeze Theorem (Pinching Theorem)

The Squeeze Theorem is a useful tool for computing limits.

Theorem Let f, g, h be functions defined on some interval $(c-\delta, c+\delta)$, for $\delta>0$, such that

$$g(x) \le f(x) \le h(x)$$

for all x in some interval $(c-\delta, c+\delta)$, except possibly at $x=c$.

If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x)$ exists and is equal to L.

For the most part, this theorem is applied when $g(x)\le f(x) \le h(x)$ for all x, but it is very useful to remember the same statement holds when $g(x)\le f(x)\le h(x)$ only for x near c.

Example 1) $\lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right) = 0$

Note that for all $x \ne 0$,

$$-1 \le \sin\left(\tfrac{1}{x}\right) \le 1.$$

Therefore,

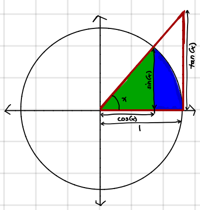
$$-x^2 \le x^2\sin\left(\tfrac{1}{x}\right) \le x^2.$$

Since $\lim_{x\to0}(-x^2) = \lim_{x\to 0} x^2 = 0$, we must have that

$$\lim_{x\to 0}\left(x^2\sin\left(\tfrac{1}{x}\right)\right) = 0.$$

2) $\lim_{x\to 0}\frac{\sin x}{x} = 1.$

Proof. Consider the unit circle given below and the ray making an angle of x radians with the x-axis $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$.

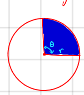


Let $A(x)$ = area of large red triangle $= \frac{1}{2}\tan(x)$

$a(x)$ = area of small green triangle. $= \frac{1}{2}\sin(x)\cos(x)$

$s(x)$ = area of sector (blue + green region) $= \frac{1}{2}x$

The area of a sector of a circle of radius r, with angle θ, is $A = \frac{1}{2}r\theta$.



Due to each region containing a smaller one, we have

$$a(x) \le s(x) \le A(x) \qquad \text{for } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}$$

$$\Longrightarrow \quad \tfrac{1}{2}\sin(x)\cos(x) \le \tfrac{1}{2}x \le \tfrac{1}{2}\tan(x) \qquad -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}$$

Dividing by $\sin(x)$, which is possible unless $x=0$,

$$\tfrac{1}{2}\cos(x) \le \tfrac{1}{2}\frac{x}{\sin(x)} \le \tfrac{1}{2}\frac{1}{\cos(x)}$$

Multiplying by 2,

$$\cos(x) \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)}$$

Taking reciprocals,

$$\frac{1}{\cos(x)} \ge \frac{\sin(x)}{x} \ge \cos(x)$$

As $x\to 0$, $\cos(x)\to 1$ and $\frac{1}{\cos(x)} \to 1$. Therefore, $\frac{\sin(x)}{x} \to 1$ and $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$.

3) From $\lim_{x\to0}\frac{\sin x}{x} = 1$, we also obtain that

$$\lim_{x\to 0}\frac{1-\cos(x)}{x} = \lim_{x\to 0}\frac{1-\cos(x)}{x}\cdot\frac{1+\cos(x)}{1+\cos(x)}$$

$$= \lim_{x\to 0}\frac{1-\cos^2(x)}{x(1+\cos(x))}$$

$$= \lim_{x\to 0}\frac{\sin^2(x)}{x(1+\cos(x))}$$

$$= \lim_{x\to 0}\left[\frac{\sin(x)}{x}\cdot\frac{\sin(x)}{1+\cos(x)}\right]$$

$$= \left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)\cdot\left(\lim_{x\to 0}\frac{\sin(x)}{1+\cos(x)}\right)$$

$$= 1\cdot 0$$

$$= 0$$

4) We have the follow set of inequalities

$$0 \le x^2 \le |x| \qquad \text{for } -1 \le x \le 1.$$

Taking $\lim_{x\to 0}$, we have that $\lim_{x\to 0} 0 = \lim_{x\to 0}|x| = 0$. Therefore, by the Squeeze Theorem, $\lim_{x\to 0} x^2 = 0$.

Obviously, we could evaluate $\lim_{x\to 0} x^2$ without the Squeeze Theorem, but this is a good illustration for the theorem when we know all the quantities involved.

We have the following list of limits which should be memorized:

Theorem We have

1) $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$

2) $\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$

3) $\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \frac{1}{2}$.

Example 1) $\lim_{x\to 0}\frac{\sin(3x)}{4x} = \lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot\frac{3}{4}$

$$= \frac{3}{4}\cdot\underbrace{\lim_{x\to 0}\frac{\sin(3x)}{3x}}_{=1}$$

$$\boxed{\lim_{x\to 0}\frac{\sin(3x)}{3x} = \frac{3}{4}}$$

2) $\lim_{x\to 0}\frac{1-\cos(4x)}{25x^2} = \underbrace{\lim_{x\to 0}\frac{1-\cos(4x)}{16x^2}}_{=1/2}\cdot\frac{16}{25}$

$$= \frac{1}{2}\cdot\frac{16}{25}$$

$$\lim_{x\to 0}\frac{1-\cos(4x)}{25x^2} = \frac{8}{25}$$

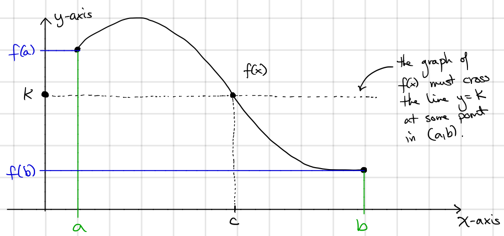
Intermediate Value Theorem and Extreme Value Theorem

Intuitively, the Intermediate Value Theorem says that a continuous function does not "jump" from one value to another

Theorem (IVT) Let f be a continuous function on $[a,b]$.
For any value K (strictly) between $f(a)$ and $f(b)$, there is some c such that $a<c<b$ and $f(c)=K$.

Notes: "there is some c" means there is at least one such point; there may be more

Picture:



Another way of phrasing the IVT is that whenever K is strictly between $f(a)$ and $f(b)$, the equation $f(x)=K$ has a solution $x=c$ in the interval (a,b).

Example 1) There is a solution to the equation $x^5-x+1=0$.
Let $f(x)=x^5-x+1$. Then, polynomial $\Rightarrow$ continuous everywhere. In particular, f is continuous on $[-2,1]$.
Now, $f(-2)=-29$ and $f(1)=1$.
Since $K=0$ lies between -29 and 1, the IVT says there is some c such that
$-2<c<1$ and $f(c)=c^5-c+1=0$.
Therefore, c is a solution of $x^5-x+1=0$.

2) Let $f(x)=\frac{1}{x}$.
Then, $f(-1)=\frac{1}{-1}=-1$ and
$f(1)=\frac{1}{1}=1$.
Now, 0 lies between -1 and 1 but there does not exist any c such that $\frac{1}{c}=0$. Why?
Because $f(x)=\frac{1}{x}$ is not continuous on $[-1,1]$.

3) The equation $2\cos(x)-x+1=0$ has a solution in $[1,2]$.

A second very fundamental theorem about continuous functions is

Theorem (Extreme-Value Theorem) Let f be continuous on $[a,b]$. Then
1) f attains a maximum on $[a,b]$, i.e. there is some c in $[a,b]$ such that
$f(x)\le f(c)$ for all x in $[a,b]$.
2) f attains a minimum on $[a,b]$, i.e. there is some c in $[a,b]$ such that
$f(x)\ge f(c)$ for all x in $[a,b]$.

The maximum and minimum values of f are together called the extreme values of f.

Example 1) The function $f(x)=\frac{1}{x}$ is continuous on $(0,\infty)$, but it does not attain its extreme values.

2) The function $f(x)=x$ defined and continuous on $(0,1)$ does not attain its maximum or minimum values.
This shows that $f(x)$ must be defined (and continuous) on a closed, bounded interval $[a,b]$

3) The function $f(x)=x^3-3x+2$ is continuous on $[-2,1]$. Therefore, it must attain its maximum and minimum values:
Maximum: $f(-1)=4$
Minimum: $f(-2)=f(1)=0$
Therefore, extremum values can be attained at two different points.
Extreme values can also be attained at the endpoints of $[a,b]$.

Finding extreme values will be one of the main applications for derivatives.

Example The function $f(x)=|x|$ is continuous on $[-1,1]$.
It attains its maximum at the endpoints $x=1$ and $x=-1$. Maximum $=1$
It attains its minimum at $x=0$. Minimum $=0$.