Handout

30 November 2018

1. Find the minimal distance between the line

$$y = 2x + 3$$

and the point $(0, 1)$.

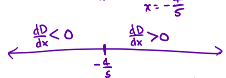
$$D = (x-0)^2 + (y-1)^2$$
$$= x^2 + (2x+3-1)^2$$
$$= x^2 + (2x+2)^2$$
$$= x^2 + 4(x^2+2x+1)$$
$$D = 5x^2 + 8x + 4$$

Want to minimize D.

1) No singular pts

2) Stationary pts:

$$\frac{dD}{dx} = 10x + 8 = 0$$
$$x = -\frac{4}{5}$$



$x = -\frac{4}{5}$ is the global minimum of D.

At $x = -\frac{4}{5}$, $D = 5(-\frac{4}{5})^2 + 8(-\frac{4}{5}) + 4$

$$= \frac{16}{5} - \frac{32}{5} + \frac{20}{5}$$

$$\boxed{D = \frac{4}{5}}$$

2. Find the point on

$$y = x^2$$

of minimal distance to the point $(3, 0)$.

$$D = \sqrt{(x-3)^2 + y^2} > 0$$
$$= \sqrt{(x-3)^2 + (x^2)^2}$$
$$= \sqrt{x^4 + x^2 - 6x + 9}$$
$$\frac{dD}{dx} = \frac{4x^3 + 2x - 6}{2\sqrt{x^4+x^2-6x+9}}$$
$$= \frac{2x^3 + x - 3}{\sqrt{x^4+x^2-6x+9}}$$

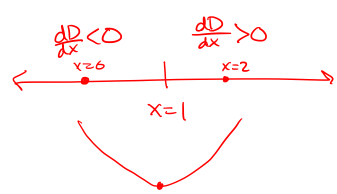
Singular: None since $(x-3)^2 + x^4 > 0$.

Stationary:

$$\frac{2x^3 + x - 3}{\sqrt{x^4+x^2-6x+9}} = 0$$

$$2x^3 + x - 3 = 0$$

$x = 1$ is a solution. There are no others since $\frac{d}{dx}(2x^3+x-3) = 6x^2+1 > 0$ and the function is always increasing



$x=1$ is global minimum.

$$D = \sqrt{x^4 + x^2 - 6x + 9}$$

$$D(1) = \sqrt{1+1-6+9}$$
$$D(1) = \boxed{\sqrt{5}}$$

3. Find the point on

$$y = \frac{1}{8}x^2$$

of minimal distance to the point $(0, 6)$.

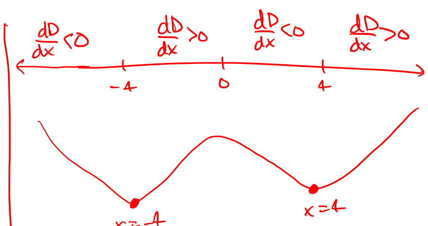
$$D = \sqrt{(x-0)^2 + (y-6)^2}$$
$$= \sqrt{x^2 + (\tfrac{1}{8}x^2 - 6)^2}$$
$$\frac{dD}{dx} = \frac{2x + (\frac{1}{4}x)\cdot 2(\frac{1}{8}x^2 - 6)}{2\sqrt{x^2 + (\frac{1}{8}x^2-6)^2}}$$
$$= \frac{2x + \frac{1}{16}x^3 - 3x}{2\sqrt{x^2 + (\frac{1}{8}x^2-6)^2}}$$
$$= \frac{\frac{1}{16}x^3 - x}{2\sqrt{x^2 + (\frac{1}{8}x^2-6)^2}}$$

Singular: None since $x^2 + (\frac{1}{8}x^2 - 6)^2 > 0$.

Stationary: $\frac{dD}{dx} = 0$

$$\frac{1}{16}x^3 - x = 0$$
$$x^3 - 16x = 0$$
$$x(x-4)(x+4) = 0$$

$x = 0$, $x = 4$, $x = -4$.

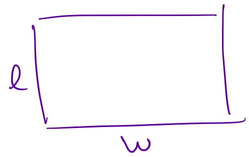


Global minimum occurs at either $x = 4$ or $x = -4$

$$D(4) = \sqrt{16 + (2-6)^2} = \sqrt{32}$$
$$D(-4) = \sqrt{32}$$

$\Rightarrow$ Minimal Distance is $\sqrt{32}$.

4. Find the dimensions of a rectangle of perimeter 24 that has the largest area.



$A = lw$

$P = 2l + 2w = 24$

$l + w = 12$

$l = 12 - w$

$A = lw$
$= w(12 - w)$
$= 12w - w^2$

Since $l, w \geq 0$, we have that w lies in the interval $[0, 12]$.

1) Singular Points: None
2) Stationary Points:

$\frac{dA}{dw} = 12 - 2w = 0$

$\Rightarrow w = 6$

3) Endpoints: $w = 0, 12$

$A(0) = 0$

$A(12) = 12(12) - 12^2$
$= 0$

$A(6) = 12(6) - 6^2$
$= 72 - 36$
$= 36$

The maximum area is $\boxed{36}$

5. Find the dimensions of a rectangle of area A that has minimal perimeter.

$A = lw \Longrightarrow l = \frac{A}{w}$ where A is constant.

$P = 2l + 2w$
$= \frac{2A}{w} + 2w$

Singular: None
Stationary: $\frac{dP}{dw} = 0$

$-\frac{2A}{w^2} + 2 = 0$

$2 = \frac{2A}{w^2}$

$w = \sqrt{A}$

Endpoints: First, $w \geq 0$.
The other endpoint occurs when $w = \infty$. at which point, $P = \infty$. This cannot be a minimum.

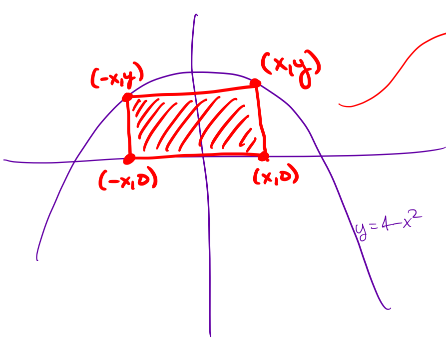
$P(0) = \infty$

$P(\infty) = \infty$

$P(\sqrt{A}) = \frac{2A}{\sqrt{A}} + 2\sqrt{A}$
$= 4\sqrt{A}$

The minimal perimeter is $\boxed{4\sqrt{A}}$

6. Find the largest possible area for a rectangle with base on the x-axis and upper vertices on the curve $y = 4 - x^2$.



$A = (2x)y = 2xy$ where $x, y \geq 0$.

$y = 4 - x^2 \rightsquigarrow A = 2x(4 - x^2)$
$A = 8x - 2x^3$

Want to maximize $A = 8x - 2x^3$.

Singular: None
Stationary: $\frac{dA}{dx} = 8 - 6x^2 = 0$

$6x^2 = 8$

$x^2 = \frac{4}{3}$

$x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$

Endpoints: The rectangle must have $x, y \geq 0$ so

$y \geq 0 \Rightarrow 4 - x^2 \geq 0$

$\Rightarrow x^2 \leq 4$

$\Rightarrow x \leq 2$

So endpoints are $x = 0$, $x = 2$.

$A(0) = 2(0)(4 - 0^2) = 0$

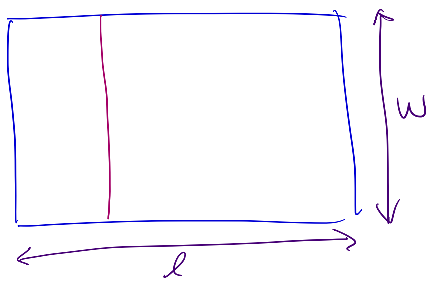
$A(2) = 2(2)(4 - 2^2) = 0$

$A(\frac{2}{\sqrt{3}}) = 2(\frac{2}{\sqrt{3}})(4 - (\frac{2}{\sqrt{3}})^2)$
$= \frac{4}{\sqrt{3}}(4 - \frac{4}{3})$
$= \frac{4}{\sqrt{3}}(\frac{8}{3})$
$= \frac{32}{3\sqrt{3}}$

$-\frac{2}{\sqrt{3}}$ does not lie in $[0, 2]$

Maximal area is $\boxed{\frac{32}{3\sqrt{3}}}$

7. A rectangular warehouse will have 5000 square feet of floor space and will be separated into two rectangular rooms by an interior wall. The cost of the exterior walls is \$150 per linear foot and the cost of the interior wall is \$100 per linear foot. Find the dimensions that will minimize the cost of building the warehouse.



Floor Space $= lw = 5000$

Cost $= C = 150(2w+2l) + 100w$

$= 400w + 300l$

$lw = 5000$

$l = \frac{5000}{w}$

$C = 400w + 300l$

$C = 400w + \frac{1500000}{w}$

1) Singular point: $w=0$

2) Stationary point:

$\frac{dC}{dw} = 400 - \frac{1500000}{w^2} = 0$

$400 = \frac{1500000}{w^2}$

$w^2 = \frac{15000}{4}$

$w^2 = 3750$

$w = \sqrt{3750}$

$= \sqrt{25 \cdot 25 \cdot 6}$

$w = 25\sqrt{6}$

We need $w > 0$

$\frac{dC}{dw} < 0$ $\frac{dC}{dw} > 0$

0 $25\sqrt{6}$ w

$w = 25\sqrt{6}$ is the global minimum.

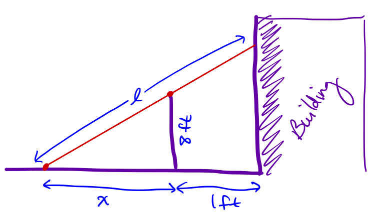
Minimal Cost is $400(25\sqrt{6}) + \frac{1500000}{25\sqrt{6}}$

$= 10000\sqrt{6} + \frac{60000}{\sqrt{6}}$

$= 10000\sqrt{6} + \frac{6 \cdot 10000}{\sqrt{6}}$

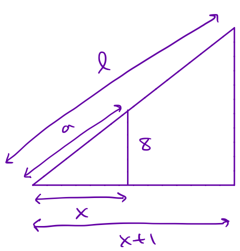
$= \boxed{20000\sqrt{6}}$

8. An 8-foot-high fence is located 1 foot from a building. Determine the length of the shortest ladder that can be leaned against the building and touch the top of the fence.



We wish to minimize l.

The distance x is > 0.



Similar Triangles: $\frac{l}{a} = \frac{x+1}{x}$

and $a = \sqrt{x^2+64}$

Therefore, $l = \frac{x+1}{x}\sqrt{x^2+64}$

$l = \left(1+\frac{1}{x}\right)\sqrt{x^2+64}$

1) No singular points

2) Stationary points:

$\frac{dl}{dx} = \left(-\frac{1}{x^2}\right)\sqrt{x^2+64} + \left(1+\frac{1}{x}\right)\frac{2x}{2\sqrt{x^2+64}} = 0$

$-\frac{1}{x^2}(x^2+64) + \left(1+\frac{1}{x}\right)x = 0$

$-1 - \frac{64}{x^2} + x + 1 = 0$

$x - \frac{64}{x^2} = 0$

$x^3 = 64$

$x = 4.$

$\frac{dl}{dx} < 0$ $\frac{dl}{dx} > 0$

0 $x=4$

$\frac{dl}{dx}$ at $x=1$ is $(-1)\sqrt{65} + (2)\frac{2}{2\sqrt{65}}$

$= \frac{2\sqrt{65}}{65} - \sqrt{65} < 0$

$\frac{dl}{dx}$ at $x=5$ is

$-\frac{1}{25}\sqrt{25+64} + \left(1+\frac{1}{5}\right)\frac{5}{\sqrt{25+64}}$

$= -\frac{\sqrt{89}}{25} + \frac{6}{5}\cdot\frac{5}{\sqrt{89}}$

$= \frac{6\sqrt{89}}{89} - \frac{\sqrt{89}}{25} > 0$

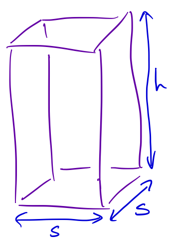
Therefore, $x=4$ is the global minimum

$l = \left(1+\frac{1}{4}\right)\sqrt{4^2+64}$

$= \frac{5}{4}\sqrt{80} = \frac{5}{4}\sqrt{16 \cdot 5}$

$\boxed{l = 5\sqrt{5}}$

9. What is the maximum volume for a rectangular box (square base, no top) made from 12 square feet of cardboard?



$V = s^2 h$

$S = 2s^2 + 4sh$

$S = 12$

$2s^2 + 4sh = 12$

$s^2 + 2sh = 6$

$2sh = 6 - s^2$

$h = \frac{6 - s^2}{2s}$

$V = s^2 \cdot \frac{6 - s^2}{2s}$

$V = \frac{1}{2} s (6 - s^2)$

$\frac{dV}{ds} = \frac{1}{2}(6 - s^2) + \frac{1}{2} s (-2s)$

$\frac{dV}{ds} = \frac{1}{2}(6 - s^2) - s^2$

$= 3 - \frac{3}{2} s^2 = 0$

$s^2 = 2$

$s = \sqrt{2}$

$h = \frac{6 - (\sqrt{2})^2}{2\sqrt{2}}$

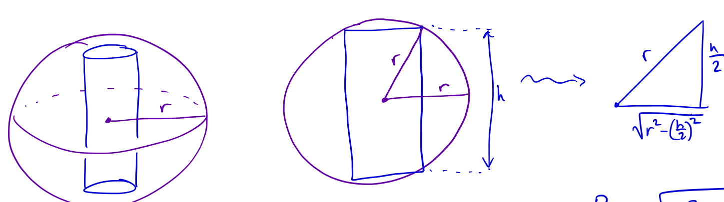
$= \frac{4}{2\sqrt{2}}$

$h = \sqrt{2}$

$V = (\sqrt{2})^3$

$\boxed{V = 2\sqrt{2}}$

10. A right circular cylinder is inscribed in a sphere of radius r. Find the dimensions of the cylinder that maximize the volume of the cylinder.



If the cylinder has height h, then it has radius $R = \sqrt{r^2 - \frac{1}{4}h^2}$.

The height h lies in $[0, 2r]$.

The volume V of the cylinder is

$V = \frac{1}{2}\pi R^2 h$

$= \frac{1}{2}\pi (r^2 - \frac{1}{4}h^2) h$

$= \frac{1}{2}\pi r^2 h - \frac{1}{8}\pi h^3$

Maximize V:

1) No singular points

2) Stationary points:

$\frac{dV}{dh} = \frac{1}{2}\pi r^2 - \frac{3}{8}\pi h^2 = 0$

$\frac{1}{2}\pi r^2 = \frac{3}{8}\pi h^2$

$\frac{4}{3} r^2 = h^2$

$h = \frac{2r}{\sqrt{3}}$

Endpoints:

$h = 0$

$h = 2r$

$V(0) = 0$

$V(2r) = 0$

$V\left(\frac{2r}{\sqrt{3}}\right) = \frac{1}{2}\pi\left(r^2 - \frac{1}{4}\left(\frac{4r^2}{3}\right)\right)\frac{2r}{\sqrt{3}}$

$= \frac{1}{2}\pi\left(\frac{2}{3}r^2\right)\frac{2r}{\sqrt{3}}$

$= \frac{2\pi}{3\sqrt{3}} r^3$

Maximum Volume $= \boxed{\frac{2\pi}{3\sqrt{3}} r^3}$